Universal Signatures of Quantum Chaos

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We discuss fingerprints of classical chaos in spectra of the corresponding bound quantum systems. A novel quantity to measure *quantum chaos in spectra* is proposed and a conjecture about its universal statistical behavior is put forward. Numerical as well as theoretical evidence is provided in favor of the conjecture.

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In classical physics *chaos* can be characterized by the long-time behavior of the dynamics, the most obvious property being a sensitive dependence on initial conditions. Time correlations of classical observables decay (possibly exponentially) for $t \to \infty$ when the system shares the mixing property, reflecting a complete loss of information on the system. Integrable systems, however, show quasiperiodic time evolutions which result in nondecreasing time correlations. For bound conservative systems, which we are exclusively considering in the sequel, the quantum mechanical time evolution is almost periodic. This is due to the discrete spectrum of the time-evolution operator $U(t) = e^{-\frac{i}{\hbar}Ht}$, where H denotes the quantum Hamiltonian with (discrete) spectrum $\{E_n, n \in \mathbf{N}\}$. Thus the time-correlation function of two states ψ, φ from the quantum mechanical Hilbert space reads $\langle \psi, U(t)\, \varphi \rangle = \sum_{n=1}^{\infty} c_n\, e^{-\frac{i}{\hbar}E_n t}$. For $t \to \infty$ this neither increases nor decreases but rather fluctuates perpetually, irrespective of the integrable or chaotic nature of the classical limit. There thus exists no *quantum chaos (QC)* that manifests itself in the long-time behavior of the dynamics.

Instead, one could consider the limit $t = \infty$ in quantum mechanics and would thus study properties of stationary states, that is, of eigenvalues and eigenfunctions of H. If one were to identify unique fingerprints of the corresponding classical dynamics in properties of stationary quantum states, one could use these to define QC. Ideally, classically chaotic systems should be characterized by a random behavior of these fingerprints that qualify the systems to be called "chaotic" also in quantum mechanics.

The present perception of spectral statistics in QC asserts that quantum energy spectra of individual systems with even a low number of degrees of freedom (≥ 2) can be described by the results of random matrix theory (RMT) if only the classical limit is strongly chaotic (a K system). In contrast, statistical properties of quantum energy spectra of classically integrable systems should be described by Poissonian random processes. So far this characterization of different types of spectral fluctuations has been a purely phenomenological one; it is, however, desirable to find a theoretical justification of the results which makes use of the properties of the corresponding classical systems and which would allow one to predict the spectral statistics once the classical dynamics is known. Using periodic-orbit theory [1], Berry and Tabor [2] analyzed the level spacings of classically integrable systems, and later Berry [3] extended the analysis to the spectral rigidity for both integrable and chaotic systems. In this way the above assertions on spectral statistics were confirmed for "generic" systems on small and medium scales in the spectra. Berry, however, obtained a saturation of the two-point statistics on large scales in contradiction to the predictions based on RMT; long-range correlations occur due to the relevance of the fine structure in the periods of short periodic orbits.

In addition, a class of strongly chaotic systems was found [4] for which the traditional measures of spectral fluctuations, i.e., the level spacings distribution and the two-point statistics, nearly behave as is expected for classically integrable systems. This phenomenon occurs for geodesic flows on hyperbolic surfaces, i.e., Riemannian surfaces with metrics of constant negative Gaussian curvatures, whose fundamental groups are of an arithmetical origin [5]; thus the notion of *arithmetical chaos* was introduced. It was observed that for these systems a crucial assumption in Berry's periodic-orbit analysis is strongly violated in that the arithmetical systems are exceptional in their classical dynamics by exponentially growing multiplicities of lengths of periodic orbits [56]. It was, however, possible to devise a periodic-orbit analysis in the spirit of Berry that explains the observed peculiarities of the spectral statistics [7].

The example of arithmetical chaos clearly teaches us that there do not exist universal fingerprints of classical chaos that manifest themselves in the level spacings distribution or in the two-point statistics. Moreover, the expectation based on RMT does not really provide a criterion in which chaotic systems distinguish themselves in a particular randomness, since the spectral fluctuations expressed by the considered quantities are stronger (Poissonian-like) in the integrable case. It thus seems desirable to introduce a quantity that clearly distinguishes quantum systems with chaotic classical limits from those with integrable ones, and which in a more direct and intuitive way expresses the random character of spectral fluctuations in the former case. It is the aim of this Letter to put forward a conjecture on a suitable quantity to measure QC in spectra

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and to provide evidence in favor of it. A preliminary announcement can be found in [8].

In the following we will discuss bound conservative systems of $f \geq 2$ degrees of freedom with actions $S_\gamma(E) = \int_\gamma \mathbf{p} \cdot d\mathbf{q}$ of classical periodic orbits γ that scale according to $S_\gamma(\lambda E) = \lambda^\alpha S_\gamma(E)$, $\lambda > 0$. We denote the positive root E^α by $x = E^\alpha$ so that $S_\gamma(E) = (E/E_0)^\alpha S_\gamma(E_0) =: x R_\gamma$, where R_γ does not depend on the energy E. Examples of scaling systems are billiards and geodesic flows on Riemannian manifolds, where $x = p = \sqrt{E}$ is the modulus of the momentum (in suitable units) and $R_\gamma = l_\gamma$ is the geometrical length of γ. Furthermore, Hamiltonian systems with scaling potentials, $V(\lambda\mathbf{q}) = \lambda^\kappa V(\mathbf{q})$, $\lambda > 0$, yield $\alpha = \frac{1}{2} + \frac{1}{\kappa}$. The discrete spectrum $\{E_n; n \in \mathbf{N}\}$ of the Hamiltonian H will be studied in terms of the variable x, i.e., in the form $\{x_n = E_n^\alpha; n \in \mathbf{N}\}$. Its *spectral staircase* reads $\mathcal{N}(x) := \#\{x_n; 0 \leq x_n \leq x\}$, and can in general be decomposed into a smooth part $\overline{\mathcal{N}}(x)$ describing a "mean behavior" of $\mathcal{N}(x)$, and a fluctuating part $\mathcal{N}_{\rm fl}(x)$.

One can now express $\mathcal{N}_{\rm fl}(x)$ for strongly chaotic systems, i.e., K systems whose periodic orbits are isolated and unstable, by the *dynamical zeta function* which reads for $f = 2$ and $\mathrm{Re}\, s > \sigma_a$

$$Z(s) = \prod_\gamma \prod_{n=0}^{\infty} \left(1 - \chi_\gamma \sigma_\gamma^n e^{-[sR_\gamma + (n+\frac{1}{2})u_\gamma]}\right), \qquad (1)$$

as it arises from Gutzwiller's semiclassical trace formula [1]. $Z(s)$ is a function of the variable $s = -ix$, where the outer product in Eq. (1) runs over all primitive periodic orbits γ of the classical system and u_γ denotes the stability exponent of γ; χ_γ is a phase factor attached to γ, and $\sigma_\gamma = \pm 1$ depends on whether γ is a direct hyperbolic ($\sigma_\gamma = +1$) or an inverse hyperbolic ($\sigma_\gamma = -1$) orbit; $\sigma_a > 0$ denotes the abscissa of absolute convergence (the *entropy barrier*) of the Euler product (1). Using Gutzwiller's trace formula one obtains that in the semiclassical limit the scaled eigenvalues x_n are given by the zeros of $Z(s)$ at $s_n = \pm i x_n$. The spectral staircase $\mathcal{N}(x)$ thus counts the number of zeros of $Z(s)$ on the *critical line* $s = -ix$, $x \in \mathbf{R}$, in the interval $[0, x]$. Once $Z(s)$ is holomorphic in a strip $|\mathrm{Re}\, s| \leq \delta$, $\delta > 0$, the argument principle, which is a common tool in analytic number theory, yields that

$$\mathcal{N}_{\rm fl}(x) = \frac{1}{\pi} \arg Z(ix)\,, \qquad (2)$$

and $\overline{\mathcal{N}}(x)$ is such that $Z(ix)\, e^{i\pi\overline{\mathcal{N}}(x)}$ is real valued for $x \in \mathbf{R}$, implying the *functional equation* $Z(ix)\, e^{i\pi\overline{\mathcal{N}}(x)} = Z(-ix)\, e^{-i\pi\overline{\mathcal{N}}(x)}$ [9]. We remark that the assertion on the holomorphy of $Z(s)$ can in general not be proven rigorously, but is known to be true for geodesic flows on hyperbolic manifolds where $Z(s)$ is given by *Selberg's zeta function* [10]. In other cases Gutzwiller's trace formula suggests that Eq. (2) holds in the semiclassical limit. We stress that for classically integrable systems a representation (2) of $\mathcal{N}_{\rm fl}(x)$ in terms of a zeta function does not exist.

We now suppose that the asymptotic behavior of the spectral staircase is given by $\mathcal{N}(x) \sim \overline{\mathcal{N}}(x) \sim c\, x^\beta$, $x \to \infty$, with some positive constants c and β. In the case of billiards or geodesic flows and for scaling potentials this holds with $\beta = f$. The mean spectral density then behaves as $\overline{d}(x) = \frac{d\overline{\mathcal{N}}(x)}{dx} \sim \beta c x^{\beta - 1}$ for $x \to \infty$. The *spectral rigidity* of $\{x_n; n \in \mathbf{N}\}$ is defined as

$$\Delta_3(L; x) :=$$

$$\left\langle \min_{(A,B)} \frac{\overline{d}(x)}{\beta L} \int_{-\frac{\beta L}{2\overline{d}(x)}}^{+\frac{\beta L}{2\overline{d}(x)}} d\varepsilon \left[\mathcal{N}(x + \varepsilon) - A - B\varepsilon\right]^2 \right\rangle, \qquad (3)$$

where $\langle \cdots \rangle$ denotes an average in x over an interval $[x - \delta, x + \delta]$ with $\overline{d}(x)^{-1} \ll \delta \ll x$. In the limit $L \to \infty$ and $x \to \infty$ such that $x/l = 2x\overline{d}(x)/\beta L \to \infty$ one obtains that [7]

$$\Delta_\infty(x) \sim \left\langle \frac{1}{2l} \int_{x-l}^{x+l} d\varepsilon\, \mathcal{N}_{\rm fl}(\varepsilon)^2 \right\rangle, \qquad (4)$$

where $\Delta_\infty(x) = \lim_{L\to\infty} \Delta_3(L; x)$. Thus $\Delta_\infty(x)$ approaches for $x \to \infty$ the second moment of the distribution of the values of $\mathcal{N}_{\rm fl}(x)$. However, in all interesting cases $\Delta_\infty(x)$ diverges for $x \to \infty$ so that Eq. (4) suggests to define the quantity

$$W(x) := \frac{\mathcal{N}_{\rm fl}(x)}{\sqrt{\Delta_\infty(x)}}, \qquad (5)$$

whose limit distribution (if it exists) has for $x \to \infty$ a second moment of 1. Since by definition $\mathcal{N}_{\rm fl}(x)$ describes the fluctuations of the spectral staircase about a mean behaviour, the first moment of $W(x)$ vanishes [7]. We thus conclude that the distribution

$$\frac{1}{2l} \mu\{\varepsilon \in [x - l, x + l]; W(\varepsilon) \in [a, b]\}$$

$$= \frac{1}{2l} \int_{x-l}^{x+l} d\varepsilon\, \chi_{[a,b]}(W(\varepsilon)) \qquad (6)$$

of $W(\varepsilon)$ on the interval $[x - l, x + l]$ has in the limit $x, l \to \infty$ mean zero and unit variance. In Eq. (6) $\chi_{[a,b]}(w)$ denotes the characteristic function of the interval $[a, b]$, and μ is the Lebesgue measure on $\mathbf{R}$.

We are now in a position to formulate our conjecture: *For bound conservative and scaling systems the quantity $W(x)$, Eq. (5), possesses a limit distribution for $x \to \infty$ with zero mean and unit variance. This distribution is absolutely continuous with respect to Lebesgue measure on the real line with a density $f(w)$. Thus*

$$\lim_{x,l\to\infty} \frac{1}{2l} \int_{x-l}^{x+l} d\varepsilon\, \chi_{[a,b]}(W(\varepsilon)) = \int_a^b dw\, f(w). \qquad (7)$$

Furthermore,

$$\int_{-\infty}^{+\infty} dw\, w\, f(w) = 0, \quad \int_{-\infty}^{+\infty} dw\, w^2 f(w) = 1. \quad (8)$$

If the corresponding classical system is strongly chaotic, then $f(w)$ is a Gaussian, $f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}$. In contrast, a classically integrable system leads to a non-Gaussian density $f(w)$.

We want to add a few remarks: (i) The conjecture is proven for some integrable systems [11], namely, geodesic flows on flat two-dimensional tori (in electromagnetic fields of Aharonov-Bohm type). There the density $f(w)$ roughly behaves as $f(w) \sim c_1 e^{-c_2 w^4}$, $w \to \infty$. (ii) In many respects the complex zeros of the Riemann zeta function $\zeta(s)$ behave like scaled eigenvalues x_n of a hypothetical classically chaotic system without antiunitary symmetry. The analog of Eq. (2) reads $\mathcal{N}_{\rm fl}(x) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + ix)$. It has been proven using *Selberg's moment formalism* [12] that the corresponding quantity $W(x)$, with $\Delta_\infty(x) \sim \frac{1}{2\pi^2} \ln\, \ln x$, has a Gaussian limit distribution and thus is in accordance with our conjecture. Moreover, the same result has been obtained by Selberg [13] for a more general class of zeta functions. The crucial properties that had to be required for the Gaussian limit distribution to be proven was a representation as a Dirichlet series and the existence of a functional equation. Since in general dynamical zeta functions (1) can be represented by Dirichlet series and they obey a functional equation (at least in a semiclassical approximation), we are led to expect a Gaussian limit distribution for strongly chaotic systems as expressed by the conjecture. We are reminded that the existence of a functional equation is assured for geodesic flows on hyperbolic surfaces. These systems may hence be the most accessible ones for a proof of the conjecture (possibly by Selberg's moment formalism). (iii) We restricted our analysis of spectral fluctuations to scaling systems. One could expect that our conjecture extends to general bound conservative systems. The fluctuations would then have to be considered in the energy variable E itself since no other suitable variable seems available. But notice that one cannot extend the analogy to the (Riemann) zeta function in a simple manner. It should also be pointed out that so far almost exclusively scaling chaotic systems have been analyzed numerically in their spectral properties. (iv) A Gaussian limit distribution can be viewed as the validity of a *central limit theorem* for the spectral fluctuations. The *spectral entropy*

$$\mathcal{E}[f] := -\int_{-\infty}^{+\infty} dw\, f(w) \ln f(w) \quad (9)$$

measures a mean unlikelihood for $W(x)$ to be of a specific value and thus provides a quantitative measure for spectral randomness. Under the constraint of a fixed variance, which is always 1 here, $\mathcal{E}[f]$ is maximized by a normal distribution of mean zero. Thus the content of our conjecture is that *classically strongly chaotic systems have maximally random quantum spectra.*

From Berry's semiclassical analysis of the spectral rigidity [3] one obtains that for classically integrable billiards $\Delta_\infty(x) \sim c_i\, x$, $x \to \infty$, with some nonuniversal constant c_i. For rigorous results, see Ref. [14]. "Generic" classically chaotic systems with an antiunitary symmetry yield $\Delta_\infty(x) \sim [(\beta - 1)/\pi^2] \ln x$, whereas those without such a symmetry show $\Delta_\infty(x) \sim [(\beta - 1)/2\pi^2] \ln x$. In arithmetical chaos one observes [7] that $\Delta_\infty(x) \sim (A/4\pi^2) x/\ln x$, where A denotes the area of the respective arithmetic surface. This informtion thus enables one to construct the quantity $W(x)$ asymptotically for $x \to \infty$ and then to study its distribution (6) for $x, l \to \infty$.

We now provide numerical evidence in favor of our conjecture for three strongly chaotic systems. In Fig. 1 histograms of the distributions of the respective quantities $W(x)$ on certain finite intervals are presented together with Gaussian fits. In the definition (5) of $W(x)$ the saturation value $\Delta_\infty(x)$ enters which, as mentioned above, is only known asymptotically for $x \to \infty$. We observe that with our finite x values we could not pass to the asymptotic regime so that the observed distributions show variances that have not yet reached the limiting value of 1, which by construction has to be attained for $x \to \infty$. But most importantly, the Gaussian form of the distributions is already clearly statistically significant. In Fig. 1(a) we present the results for the geodesic flow on a nonarithmetic compact hyperbolic surface of genus two (see Ref. [15]) for an x interval containing the 4500th–6000th eigenvalue. A second example is provided by a billiard on the hyperbolic plane in a triangle with angles $(\pi/2, \pi/3, \pi/8)$, which is known to show arithmetical chaos. The result, based on the first 1040 eigenvalues, is shown in Fig. 1(b). Finally we present in Fig. 1(c) the result obtained from the first 1850 eigenvalues of the truncated hyperbola billiard. The billiard domain on the Euclidean plane is given by that of the desymmetrized hyperbola billiard (see Ref. [9]), truncated by a circular arc perpendicular to the x_1 axis.

So far we have been dealing with signatures of QC in spectra. The same question can, however, also be addressed to eigenfunctions of H. Already in 1977 Berry conjectured [16] that the values of eigenfunctions $\psi_n(\mathbf{q})$ would be Gaussian distributed in the semiclassical limit when the classical system is chaotic. For integrable systems eigenfunctions are known to concentrate on the invariant tori in phase space. Thus the conjecture on the statistical behavior of wave functions reads analogously to the one concerning spectra: *The distributions of the values of individual eigenfunctions of H converge for $n \to \infty$ to an absolutely continuous distribution with respect to Lebesgue measure. Once the corresponding classical system is strongly chaotic this limit distribution has a Gaussian density, whereas for classically integrable systems the density is non-Gaussian.* Several numerical

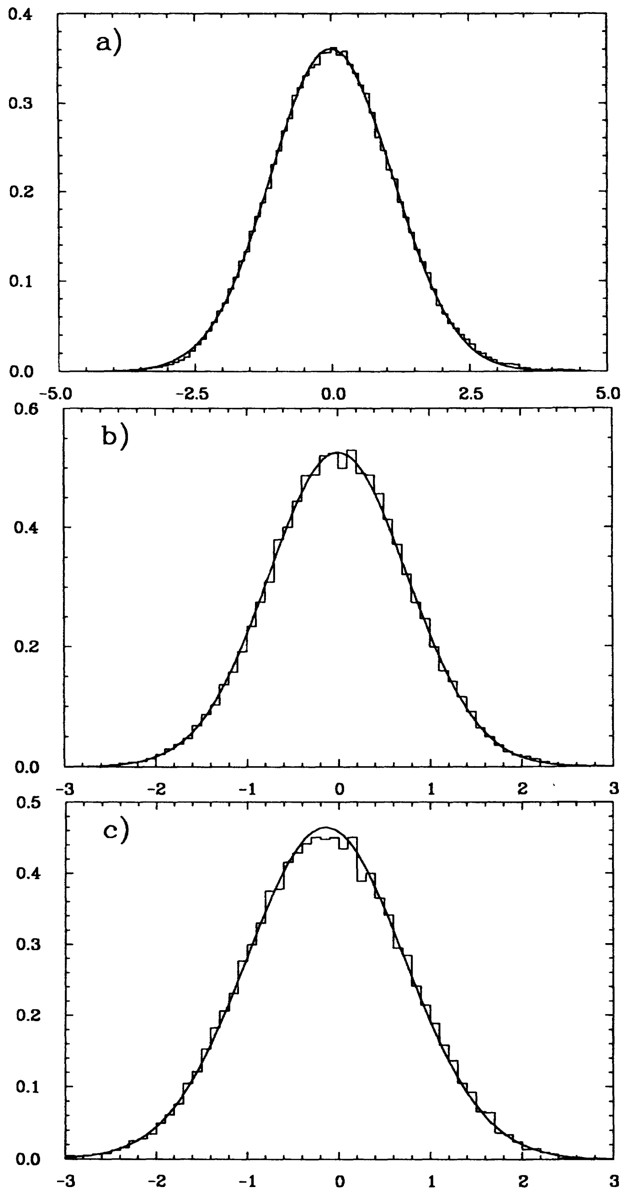

FIG. 1. The distributions of the quantity $W(x)$ are shown for the three chaotic systems as explained in the text.

tests have been performed in the past also confirming this conjecture [17].

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