Limits of Universality in Disordered Conductors

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A straightforward microscopic approach is developed to evaluate the distribution of transmission eigenvalues for a disordered phase-coherent conductor. In the diffusive limit, the calculation generalizes the result of random matrix theory that the universal distribution is not dependent on the shape of the conductor. Extended defects such as tunnel barriers and grain boundaries were shown to break the universality. If the resistance of the tunnel barrier comprises half of the total resistance, a drastic change of the distribution is found. This indicates an analog of localization transition.

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In recent years, a large body of work has appeared to reveal the basic properties of phase-coherent conductors [1]. In such a conductor, a charge carrier diffuses over the resistive region with no inelastic collision taking place. It is feasible at low temperatures in micron size junctions.

The physics of such conductors displays many universal features. The most famous example are the universal conductance fluctuations (UCF) [2]: the conductance varies from sample to sample within the value of the order of $e^2/\hbar$. It does not depend on material properties of the conductor and depends only slightly on its shape and the resistivity distribution. The less known but probably more striking example is the universal suppression of shot noise [3]. An excess current noise power in the diffusive conductor was predicted to comprise exactly *one-third* of the classical result $P = 2eI$. This is believed to depend neither on the sample shape nor its resistivity distribution. We show below that this is indeed true.

Both phenomena can be well understood using the concept of open and closed channels [4,5]. The transport properties of a conductor are completely determined by transmission matrix $\mathbf{t}$ [6]. In particular, the conductance is given by the Landauer formula

$$G = \frac{e^2}{\pi\hbar}\mathrm{Tr}(\mathbf{t}^\dagger\mathbf{t}) = \frac{e^2}{\pi\hbar}\sum_n T_n, \qquad (1)$$

where we introduce eigenvalues of transmission matrix square (transmissions) T_n corresponding to the nth eigenvector (channel). In a disordered conductor all channels seem alike, and it is natural to assume comparable transmissions for all channels.

Dorokhov [7] was the first to point out that this assumption is irrelevant. In fact, most of the channels are "closed" possessing exponentially small transmission. The transport is due to a few "open" channels having transmissions of the order of unity, and the distribution of those,

$$\rho(T) \equiv \left\langle \sum_n \delta(T - T_n) \right\rangle = \frac{\pi\hbar G}{e^2}\frac{1}{T\sqrt{1-T}}, \qquad (2)$$

is universal. The relation (2) can be proven by random matrix methods [8]. Both methods essentially exploit an assumption of a uniform quasi-one-dimensional conductor; that is, they are formally valid only if the length of the disordered region by far exceeds its width.

I strongly doubted the universality of the result (2). I had two arguments as follows. (i) Although the result may be correct for quasi-one-dimensional geometry, the experience with UCF clearly shows that in general geometry dependence persists. (ii) The result (2) seems to be valid for any disordered conductor, for any realization of disorder. Consider a realization at which a tunnel barrier partitions off the conductor. It seemed obvious that the highest transmission through the conductor cannot exceed transparency of the barrier, this being in contrast to (2) since the latter implies that many channels have transmissions close to unity. These doubts have persuaded us to tackle the problem with a straightforward microscopic technique outlined below. The advantage of the method used is that it allows one to easily take into account all the information on the geometrical shape of the conductor, the distribution, and the properties of the defects, so that the transmission matrix can be easily evaluated beyond the quasi-one-dimensional limit where random matrix methods do not work.

In the framework of the method, we will see that both arguments given above are in fact wrong. Let us first list the results obtained.

(i) The result (2) was proven to hold for a diffusive conductor of arbitrary shape and resistivity distribution, with no extended defects. The universality of the distribution can be related to the properties of the solution of the diffusion equation determining the voltage distribution in the conductor [9].

(ii) Extended defects break the universality. If the voltage distribution has discontinuities at extended defects such as tunnel barriers, grain boundaries, and interfaces, it cannot be related to the transmission distribution, the latter depending on both properties and locations of these defects.

(iii) A simple analytical result can be obtained for a single tunnel barrier which partitions off the conductor in such a way that it coincides with an equipotential surface of the voltage distribution. If the resistance of the barrier R_T is smaller than the resistance of the diffusive part of

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the conductor R_N, transmission of some channels is very close to unity. If $R_T > R_N$, a drastic change occurs: transmissions cannot exceed some maximal value $T_{\max} < 1$. The channels with $T_n \approx 1$ relate to delocalized wave functions, and do not survive at $R_N < R_T$. Therefore an analog of localization transition occurs at $R_T = R_N$.

The technique in use exhibits many similarities with the one developed to study nonequilibrium properties of superconductors [10]. The ideology is to start with microscopic formulation of the problem in terms of exact multicomponent Green functions and then subsequently derive semiclassical equations for them. From semiclassical equations one comes to an analog of the diffusion equation.

As a first step we establish the relation between Green functions and the transmission matrix amplitudes. We consider first a disordered region with two perfect leads attached. Transverse motion in the leads is quantized giving rise to discrete modes, whereas the motion along z is not quantized. The electron Green function $G^{A(R)}_{nm}(\epsilon; z, z')$, n, m being transverse mode indexes, describes an evolution of a wave packet at Fermi energy that starts at point $z(z')$. Let us take two cross sections of the leads, one far to the left and another one to the right. At the cross sections, the electron wave functions are given by asymptotics of scattered waves, so we have for transmission from left to right

$$t_{mn} = i\sqrt{v_n v_m} G^A_{mn}(z, z'), \tag{3}$$

where z, z' belong to the left and right cross sections, respectively. The conjugated matrix is expressed through G_R.

We are ready to find expressions for $\mathrm{Tr}(\mathbf{t}^\dagger \mathbf{t})^n$. They give the momenta of transmission distribution. We replace summation over the quantized modes by integration over the transverse coordinates writing the Green function in the coordinate representation (x is a three-dimensional coordinate),

$$[\epsilon \pm i\delta - \epsilon(\hat{p_1}) - U(x)] G^{R,A}(x_1, x_2) = \delta(x_1 - x_2), \tag{4}$$

$U(x)$ being random impurity potential. For the lowest order trace this yields

$$\mathrm{Tr}(\mathbf{t}^\dagger \mathbf{t}) = \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 v_1(x_1, x_2) G^A(x_2, x_3) \times v_2(x_3, x_4) G^R(x_4, x_1). \tag{5}$$

Here $v_{1(2)}(x, x')$ stands for the operator of the current through left (right) cross section. Equation (5) contains a correlator of two current operators; thus it is equivalent to the Kubo formula for conductivity. All traces of this kind possess the same operator structure, which we write symbolically as

$$\mathrm{Tr}(\mathbf{t}^\dagger \mathbf{t})^n = \mathrm{Tr}_x (v_1 G^A v_2 G^R)^n. \tag{6}$$

It seems like (6) shall depend on a choice of cross sections, but this is not true. This can be checked with (4) and follows from the conservation law for the current. Moreover, this does not depend on the shape of a cross section and whether it is in a lead or in a disordered region. In this way, one can relax the requirement of perfect leads (which is the weakest point of the Landauer formalism).

The transmission distribution will be evaluated from the generating function, a Taylor expansion of which will give the powers (6). The key idea of the method proposed is to relate this generating function to a trace of a certain multicomponent Green function in the field of a fictitious potential ζ.

This potential will couple advanced and retarded functions; therefore the resulting Green function must be a two-by-two matrix labeled by indexes $i, j = A, R$. We introduce such a Green function with the following equation (carets denote two-by-two matrices):

$$\hat{G}(x_1, x_2) = \hat{G}^{(0)}(x_1, x_2) + \int d^3x_3 d^3x_4 \hat{G}^{(0)}(x_1, x_3)[\zeta_1 v_1(x_3, x_4)\hat{\tau}^\dagger + \zeta_2 v_2(x_3, x_4)\hat{\tau}]\hat{G}(x_4, x_2). \tag{7}$$

At the left cross section the potential ζ_1 switches an advanced Green function to a retarded one whereas the potential ζ_2 switches it back at the right cross section. Here the following matrices have been used:

$$\hat{G}^{(0)} = \begin{pmatrix} G^A & 0 \\ 0 & G^R \end{pmatrix}, \quad \hat{\tau} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$G^{A,R}$ satisfying (4). It is easy to see that traces of the Taylor series of (7) in $\zeta_1\zeta_2$ will give traces of transmission matrix powers. Therefore we will use the identity

$$\int d^3x_1\, d^3x_2 v_1(x_1, x_2) \mathrm{Tr}[\hat{\tau}^\dagger \hat{G}(x_1, x_2)] = \zeta_2 \mathrm{Tr}\left(\frac{\mathbf{t}^\dagger \mathbf{t}}{1 + \zeta_1\zeta_2 \mathbf{t}^\dagger \mathbf{t}}\right) \equiv \zeta_2 F(\zeta_1\zeta_2). \tag{9}$$

The transmission distribution will be related to $F(x)$ at complex x.

Equations (7) and (4) are suitable for any disordered system. To describe the diffusive conductor, one has to derive semiclassical equations starting from (7) and (4), that is, to account for the fact that the size of the system exceeds mean free path and the mean free path is much larger than the electron wavelength. This program has been carried out a long time ago for another multicomponent Green function which describes the superconducting state. I refer the reader to [10] and present here only a final result of the derivation.

This is an effective diffusion equation for the averaged Green function in coinciding points, $\hat{G}(x,x) \equiv i\pi\nu\hat{\Lambda}(x)$, ν being the density of states near the Fermi level per one spin direction. The matrix $\hat{\Lambda}$ obeys the unitary condition $\hat{\Lambda}^2 = \hat{1}$. The diffusion equation can be written as a conservation law for matrix current,

$$\frac{\partial \hat{j}_\alpha(x)}{\partial x_\alpha} = 0, \quad \hat{j}_\alpha(x) = D(x)\hat{\Lambda}^{-1}\frac{\partial \hat{\Lambda}}{\partial x_\alpha}. \tag{10}$$

The fictitious potential may be incorporated into boundary conditions at left and right infinity:

$$\hat{\Lambda}(-\infty) = \hat{\sigma}_z, \quad \hat{\Lambda}(\infty) = \hat{S}\hat{\Lambda}(-\infty)\hat{S}^{-1},$$

where

$$\hat{S}(\zeta_1,\zeta_2) = \exp(i\zeta_1\hat{\tau}^\dagger)\exp(i\zeta_2\hat{\tau}),$$

so that the solution of (10) explicitly does not depend upon the choice of cross sections. The same is true for the value of interest (9) which can be expressed in terms of the total matrix current through an arbitrary cross section of the conductor as follows:

$$\zeta_2 F(\zeta_1\zeta_2) = i\pi\nu\,\mathrm{Tr}(\hat{\tau}^\dagger\hat{I})\hat{I} \equiv \int_S \hat{j}_\alpha(x)N_\alpha(x), \tag{11}$$

N_α being the normal vector to the cross-section surface; greek letters denote Cartesian indices. The result does not depend on the cross section since the current conserves.

At arbitrary ζ_1,ζ_2, Eq. (10) is in general a complex nonlinear matrix equation. Fortunately, the solution we are searching for can be found easily. Let us parametrize $\hat{\Lambda}$ as follows:

$$\hat{\hat{\Lambda}} = \begin{pmatrix} \cos\theta & \sin\theta/B \\ B\sin\theta & -\cos\theta \end{pmatrix}, \quad B \equiv -i\sqrt{\frac{\zeta_2}{\zeta_1(1-\zeta_1\zeta_2)}}. \tag{12}$$

Under this parametrization, the diffusion equation (10) becomes linear,

$$\hat{j}_\alpha = D(x)\frac{\partial\theta}{\partial x_\alpha}\begin{pmatrix} 0 & 1/B \\ B & 0 \end{pmatrix}, \quad \frac{\partial}{\partial x_\alpha}D(x)\frac{\partial\theta}{\partial x_\alpha} = 0. \tag{13}$$

We choose $\zeta_1 = \zeta_2 = \sin(\phi/2)$. Boundary conditions then take the simple form $\theta(-\infty) = 0, \theta(\infty) = \phi$. This is precisely the equation one has to solve in order to find the conductance of a conductor with distributed resistivity $\propto D^{-1}(x)$. Here $\theta(x)$ may be identified with the voltage distribution over a nonuniform conductor biased by potential ϕ. The density of the current is proportional to $D(x)\nabla_\alpha\theta(x)$ and the total current through any cross section is equal to the conductance,

$$\hat{I} = \begin{pmatrix} 0 & 1/B \\ B & 0 \end{pmatrix}\frac{I}{2e^2\nu}, \quad I = G\phi. \tag{14}$$

This finally yields the main result of the present work:

$$F(\phi) \equiv \mathrm{Tr}\left(\frac{\mathbf{t}^\dagger\mathbf{t}}{1-\sin^2(\phi/2)\mathbf{t}^\dagger\mathbf{t}}\right) = \frac{\pi\hbar G}{e^2}\frac{\phi}{\sin\phi}. \tag{15}$$

The distribution of transmissions can be extracted from $F(\phi)$ in a complex plane of ϕ. Using the identity $\pi\delta(x) = \mathrm{Im}(x - i0)$ we show that

$$\rho(\cosh^{-2}\mu) = \frac{\cosh^4\mu}{2\pi i}[F(2i\mu-\pi) - F(2i\mu+\pi)]. \tag{16}$$

Substitution from (15) gives the result (2) and thus confirms its validity for an arbitrary diffusive conductor. This is because the problem can be related to the solution of the equation determining the voltage distribution in the conductor.

Let us see if we are able to do this if there are extended defects setting interfaces in the conductor. To describe this, the diffusion equation (10) shall be completed by the condition at the interface. This condition should be derived microscopically, this work being done in [11]. We rewrite this in a transparent form:

$$\hat{j}_\alpha(x)N_\alpha(x) = \frac{g(x)}{2\nu e^2}[\hat{\Lambda}_1,\hat{\Lambda}_2]. \tag{17}$$

Here x belongs to the interface, N_α is a normal vector at x, $\Lambda_{1,2}$ are matrices at two sides of the interface, and $g(x)$ is nothing but a conductivity of the interface per unit area.

In terms of θ the condition reads

$$D(x)\frac{\partial\theta}{\partial x_\alpha} = \frac{g(x)}{2\nu e^2}\sin(\theta_2 - \theta_1) \tag{18}$$

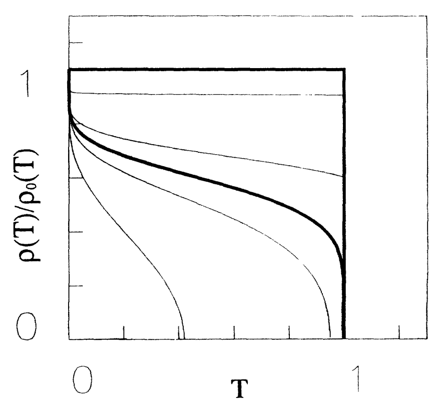

FIG. 1. Normalized distribution of transmissions for a tunnel junction in series with a diffusive conductor. The parameter G_T/G_N takes the following values starting from the lowest curve: 0.1, 0.5, 1, 2, 10. The separating curve corresponding to $G_T = G_N$ and the limiting universal distribution are drawn as thick lines.

displaying nonlinear terms. This is why the extended defects make the solution for θ different from the voltage distribution giving rise to nonuniversal $\rho(T)$.

A simple criterion can be drawn: The universality of transmission distribution is broken whenever the sharp voltage drops occur in the conductor; otherwise, it persists.

To comprehend the situation further, let us concentrate on the case when $\theta_1(x) - \theta_2(x)$ appears to be constant over the tunnel barrier interface that partitions off the conductor. Thus we have a diffusive conductor, tunnel junction, and another diffusive conductor in series and we shall match the total currents through all conductors. These currents are related to drops of θ at each element by means of either (14) or (18):

$$I = G_{N\,\mathrm{left}}\theta_1 = G_T \sin(\theta_2 - \theta_1) = G_{N\,\mathrm{right}}(\phi - \theta_2).$$

This is enough to express the current in terms of ϕ and find the function related to the distribution of transmissions [12]:

$$F(\phi) = \frac{\pi\hbar G_N}{e^2}\frac{X(\phi)}{\sin\phi},$$
$$X(\phi) + \frac{G_T}{G_N}\sin[X(\phi) - \phi] = 0. \quad (19)$$

Here G_N stands for the total conductance of two diffusive parts. To extract the distribution, we make use of (16). The answer comes in an implicit form which suffices to plot the results ($\alpha \equiv G_T/G_N$),

$$\rho(T) = f\rho_0(T), \quad T = \cosh^{-2}\mu,$$
$$\mu = \frac{1}{2}\left[\operatorname{arccosh}\left(\frac{\pi f}{\alpha\sin(\pi f)}\right) - \alpha\sqrt{\left(\frac{\pi f}{\alpha\sin(\pi f)}\right)^2 - 1}\cos(\pi f)\right].$$

We normalize the actual distribution with a universal value of $\rho_0(T)$ given by (2). In Fig. 1, the normalized distribution is plotted versus T. The distribution is always suppressed in comparison with the universal value. For a more resistive barrier, the suppression is larger. If $G_T > G_N$, the distribution remains finite at $T \to 1$, indicating that a certain fraction of the channels has almost absolute transmission. This fraction declines to zero at $G_T = G_N$. At $G_T > G_N$ the maximal transmission available cannot exceed a certain value $T_{\max}$, $T_{\max} = 1, G_N = G_T$; $T_{\max} = 4G_T^2/G_N \ll 1$, $G_T \ll G_N$. These features reveal an important new physics which is absent if tunneling occurs between two clean metals. In the latter case, transmissions through the barrier are always limited by some maximal value.

The channels with transmissions close to unity have been associated with delocalized states [5]. If the system under consideration were uniform, their disappearance at $G_N = G_T$ would have meant the true localization transition. Clearly something drastic happens at this point, but one should be cautious drawing direct conclusions. Indeed, neither resistance $R = R_T + R_N$ nor excess noise power

$$P = \frac{P_{\text{classical}}}{3}\left[1 + 2\left(\frac{R_T}{R}\right)^3\right]$$

exhibits critical behavior around the transition point for the semiclassical approach used. Such behavior would probably emerge from quantum localization corrections.

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[12] For many applications, it is more convenient to use $F(\phi)$ than the distribution itself. For example, the zero-voltage conductivity of normal-metal–superconductor boundary can be expressed as a certain integral over the distribution [13]. On the other hand, $G_{NS} = (\partial F/\partial\phi)|_{\phi=\pi/2}$. This gives $G_{NS} = G_N/[1 + \cot\mathrm{an}(X)/X], \cos(X)/X = G_N/G_T$, which coincides with the result of [14] in the relevant limit.

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