Ultraviolet-Renormalon Calculus

A. I. Vainshtein*
Theoretical Physics Institute, University of Minnesota, 116 Church Street SE, Minneapolis, Minnesota 55455
and
Budaker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

V. I. Zakharov†
University of Michigan, Ann Arbor, Michigan 48109
(Received 16 May 1994)

We consider the status of the so-called ultraviolet (UV) renormalon which contributes to large order divergences of perturbative expansions in quantum chromodynamics. We argue that although the renormalon is associated with short distance dynamics, the class of renormalon graphs is not well defined and its overall weight is not controlled by theory. From this point of view there is not much difference from the case of Borel nonsummable singularities. Phenomenologically the UV renormalon is related to an effective four-fermion interaction originating within fundamental QCD.

PACS numbers: 11.10.Gh, 11.15.Bt, 12.38.Bx

Large-order behavior of perturbative expansions has been studied for more than forty years starting from the seminal paper by Dyson [1] (for a review and collection of papers see Ref. [2]). The problem is that no matter how small the coupling constant α is, the expansion coefficients a_n grow in large orders of perturbation theory so fast that they invalidate the use of the perturbative expansion. Generically,

$$a_n \xrightarrow{n\to\infty} K n^{\gamma} \frac{n!}{S^n}, \qquad (1)$$

where K, γ, and S are constants. Because of divergences (1) there arise problems both of fundamental nature, regarding the status of perturbative expansions, and of practical importance, whether the divergences can be associated with new physical phenomena (for recent related discussion see, e.g., [3–5]).

Although the behavior (1) is of very general nature it is useful to have a particular process in mind. For the sake of definiteness we concentrate on the correlation function of two electromagnetic currents j_μ

$$\Pi_{\mu\nu} = i \int dx\, e^{iqx} \langle 0 | T\{ j_\mu(x) j_\nu(0)\} | 0\rangle$$
$$= (q_\mu q_\nu - q_{\mu\nu} q^2)\, \Pi(Q^2), \quad Q^2 = -q^2. \qquad (2)$$

The quantity $Q^2 d\Pi(Q^2)/dQ^2$ is represented as an expansion in the QCD coupling constant $\alpha_s(Q^2)$:

$$Q^2 \frac{d\Pi(Q^2)}{dQ^2} = (\text{parton model}) \times \sum_{n=0}^{n=\infty} a_n \alpha_s^n(Q^2), \qquad (3)$$

and the asymptotic of a_n is governed by Eq. (1).

What makes the picture even more complicated is that there exist different sources of the factorial growth of a_n. First, for pure combinatorial reasons the number of Feynman graphs grows with n. In this case S/α is the classical action associated with the instanton solution and [6]

$$(S)_{\text{instantons}} = 4\pi. \qquad (4)$$

Second, there exists special graphs (see Fig. 1) which receive anomalously large contribution either from very low or very large virtual momenta. These are called renormalons, infrared or ultraviolet, respectively [7–10]. Then constant S in Eq. (3) is related to the first coefficient b_0 of the β function:

$$Q^2 \frac{d}{dQ^2} \alpha_s(Q^2) = -b_0 \alpha_s^2(Q^2) - b_1 \alpha_s(Q^2) + \cdots. \qquad (5)$$

The absolute value of S is the smallest for the ultraviolet renormalon

$$(S)_{\text{UV renorm}} = -1/b_0, \quad (S)_{\text{IR renorm}} = 2/b_0, \qquad (6)$$

which means that the UV renormalon dominates at large n.

Note that, depending on sign of S, the series (3) is either sign alternating or not. This is an important point since sign oscillating series can be summed up à la Borel. Therefore only Borel nonsummable contributions are considered a real problem. The price of the nonsummability is introduction of unknown parameters. In case of the leading IR renormalon (6)—the first nonsummable singularity in QCD—this is the so-called gluon condensate [11], [10], or the matrix element $\langle 0 | \alpha_s (G^a_{\mu\nu})^2 | 0\rangle$ where $G^a_{\mu\nu}$ is the gluon field strength tensor. The corresponding uncertainty in $\Pi(Q^2)$ is of order

$$(\delta\Pi(Q^2))_{\text{IR renorm}} \sim \text{const} \times \frac{\left\langle 0 \left| \alpha_s \left(G^a_{\mu\nu}\right)^2 \right| 0 \right\rangle}{Q^4}. \qquad (7)$$

In fact, the phenomenology of the Q^{-4} terms is well developed [11]. The sound basis for this phenomenology

0031-9007/94/73(9)/1207(4)$06.00

is that the gluon condensate is numerically large and at moderate Q^2 the Q^{-4} term is not screened by lower order perturbative corrections.

In this Letter we address the ultraviolet renormalon. The status of this singularity in QCD is somewhat uncertain. On one hand, as mentioned above, numerically it should dominate at least at very large n. On the other hand, it is Borel summable and is usually disregarded for this reason. The Borel summation would turn the divergent branch of the perturbative series into a leading powerlike correction [12]:

$$(\delta\Pi(Q^2))_{\text{UV renorm}} \sim \frac{\text{const}}{Q^2}. \tag{8}$$

However, since the UV renormalon is associated with short distance dynamics which is well understood in QCD one is inclined to assume that the UV renormalon should be treated explicitly rather than phenomenologically.

We reexamine the status of the ultraviolet renormalon. Our findings can be summarized in the following way. The class of graphs contributing to asymptotic (1) with $S = -1/b_0$ is in fact ill defined. Thus the constant K in Eq. (8) presently cannot be determined even in principle [13]. From this point of view the difference between the Borel summable and Borel nonsummable cases might be rather a matter of wording. Moreover, one can argue that the UV renormalon contribution is related to matrix elements of effective four fermion interaction stemming from short distances. This observation suggests a bridge between fundamental QCD and the Nambu–Jona-Lasinio [14] kind of low-energy phenomenology.

Technically, we are mostly interested in developing means to evaluate the renormalontype graphs via the operator product expansion (OPE), which utilizes the observation that the UV renormalon is associated with virtual momenta $k \gg Q$. The idea of this OPE goes back to the paper by Parisi [9]. We will demonstrate that this OPE is indeed a practical and relatively simple way to perform calculations. The details of the calculations are given elsewhere [15].

Proceeding to the outline of the derivation, we consider a simplified model with U(1) gluonic field B_μ and N_f fermionic fields with electric charges Q_q so that the Lagrangian is

$$L = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_q \bar{q}\gamma^\mu i\partial_\mu q + gB^\mu j^1_\mu + eA^\mu j_\mu, \tag{9}$$

where g and e are the strong and electromagnetic couplings, respectively, and j^1_μ, j_μ are the gluonic and electromagnetic currents

$$j^1_\mu = \sum_q \bar{q}\gamma_\mu q, \quad j_\mu = \sum_q Q_q\bar{q}\gamma_\mu q. \tag{10}$$

We assume furthermore that $\sum Q_q = 0$ to ensure that there is no mixing between A_μ and B_μ due to fermionic loops.

The simplest renormalon-type graphs are depicted in Fig. 1 and we will explain the basic features of the technique on this example. The sum of these graphs can be cast into the form:

$$e^2\Pi_{\mu\nu}(q)\,e^{\mu}_{(1)}e^{\nu}_{(2)} = -\int \frac{d^4k}{(2\pi)^4}\frac{g^2(k^2)}{k^2}\langle\gamma^*|T|\gamma^*\rangle, \tag{11}$$

where we have made the Euclidean rotation in the integration over a gluon momentum k_μ and introduced polarization vectors $e^{\mu}_{(1,2)}$ of initial and final virtual photons. The running coupling $g^2(k^2)$ sums up vacuum bubble insertions:

$$\begin{aligned} g^2(k^2) &= 4\pi\alpha_1(Q^2)\sum_{n=0}^{\infty}[-b_0\alpha_1(Q^2)]^n \ln^n(k^2/Q^2) \\ &= g^2(Q^2)\left(1 + b_0\frac{g^2(Q^2)}{4\pi}\ln\frac{k^2}{Q^2}\right)^{-1} \end{aligned} \tag{12}$$

with $b_0 = -N_f/3\pi$. The matrix element $\langle\gamma^*|T|\gamma^*\rangle$ represents the forward amplitude of gluon-photon scattering and the operator T is

$$T = \int dx\, e^{ikx}T\left\{j^1_\mu(x), j^{1\mu}(0)\right\}. \tag{13}$$

By assumption—to be checked *a posteriori*—the momentum k flowing through the gluon line is much larger than the external momentum Q. Then it is logical to start by expanding in inverse powers of k and to consider OPE for the T product of two gluonic currents j^1_μ

$$T = \int dx\, e^{ikx}T\left\{j^1_\mu(x)j^{\mu 1}(0)\right\} = \sum c_i(k)\,O_i(0), \tag{14}$$

where O_i are local operators. Here we quote (for details see [15]) results for the OPE in the tree approximation:

$$T = -\frac{2}{3k^4}\left(e\partial^\nu F_{\nu\mu}\sum_q Q_q\bar{q}\gamma^\mu q + gD^\nu G_{\nu\mu}\sum_q \bar{q}\gamma^\mu q\right) + O(k^{-6}), \tag{15}$$

where $F_{\mu\nu}$ is the photonic field strength tensor. Note that the expansion starts from operators of dimension six.

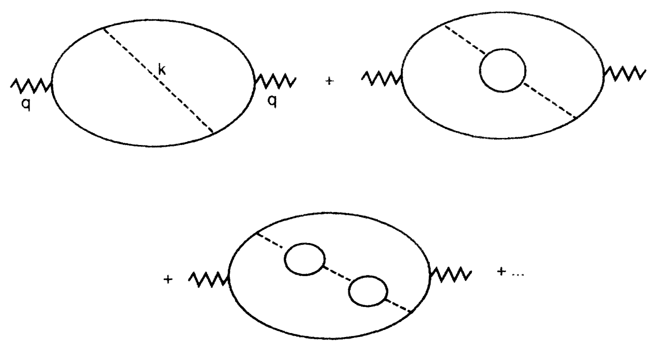

FIG. 1. Building up the simplest renormalon-type graph. Dashed line denotes gluon while solid lines refer to fermions. One starts with an exchange of a vector particle of momentum k and inserts vacuum polarization bubbles n times.

The next step is to evaluate the matrix element $\langle\gamma^*|T|\gamma^*\rangle$. The part of T containing $D^\nu G_{\nu\mu}$ contributes in fact only at the three-loop level and will be considered later. As for the part of T containing $\partial^\nu F_{\nu\mu}$ it immediately factorizes into

$$\langle\gamma^*|T|\gamma^*\rangle = -\frac{2}{3k^4}(\langle\gamma^*|e\partial^\nu F_{\nu\mu}|0\rangle\langle 0|j^\mu|\gamma^*\rangle + \langle\gamma^*|j^\mu|0\rangle\langle 0|e\partial^\nu F_{\nu\mu}|\gamma^*\rangle). \quad (16)$$

The matrix element of $\partial^\nu F_{\nu\mu}$ is trivial:

$$\langle 0|e\partial^\nu F_{\nu\mu}|\gamma^*\rangle = -e\left[q^2 e_\mu^{(1)} - q_\mu(q\cdot e^{(1)})\right], \quad (17)$$

while $\langle\gamma^*|j_\mu|0\rangle$ is given by the well-known one-loop graph and equals to

$$\langle\gamma^*|j_\mu|0\rangle = -\frac{eN_f\langle Q_q^2\rangle}{12\pi^2}\left(\ln\frac{k^2}{Q^2}\right)\left[q^2 e_\mu^{(2)} - q_\mu(q\cdot e^{(2)})\right], \quad (18)$$

where $\langle Q_q^2\rangle$ is the averaged square of electric charges. Note that the integral over the fermionic loop has been evaluated with logarithmic accuracy. The upper limit of integration, $p^2 \sim k^2$, is implied by our OPE construction while the lower bound, $p^2 \sim Q^2$, arises from account for the external momentum Q in the integrand.

Substituting the result (16) for $\langle\gamma^*|T|\gamma^*\rangle$ into (11) we come to

$$\Pi(Q^2) = \frac{N_f\langle Q_q^2\rangle}{144\pi^4}Q^2\int_{k^2\sim Q^2}^{\infty}\frac{dk^2}{k^4}\left(\ln\frac{k^2}{Q^2}\right)g^2(k^2). \quad (19)$$

We are interested in the expansion of $\Pi(Q^2)$ in $\alpha_1(Q^2)$ and resort therefore to Eq. (12). As a result there arise integrals of the type

$$Q^2\int_{k^2\sim Q^2}^{\infty}\frac{dk^2}{k^4}\ln^n\frac{k^2}{Q^2} = n!, \quad (20)$$

which are saturated by

$$(k^2)_{\rm eff} \sim Q^2 e^n. \quad (21)$$

Since at large n $k^2 \gg Q^2$, our use of OPE is justified.

Combining all the factors we arrive at

$$a_n \xrightarrow{n\to\infty} -\frac{1}{4\pi b_0}(-b_0)^n n!. \quad (22)$$

This result coincides with explicit calculations of the renormalon graph of Ref. [16]. It is worth emphasizing that the advantage of the operator product expansion is not only the compactness of the calculation but also explicit gauge invariance so that the generalization to the QCD case is straightforward and reduces to a change in b_0. Indeed, since the operator product expansion is based on a set of gauge invariant operators the only dependence on $\ln k^2/Q^2$ arises through the use of Eq. (12) (or its two-loop generalization) and we even do not need even to specify the gauge fixing or the class of graphs involved explicitly. This technical point is especially important in the case of non-Abelian gauge theories.

The OPE can also be applied to three-loop skeleton graphs, whose examples are shown in Fig. 2. One we insert vacuum bubbles into gluon lines, we get graphs with two renormalon chains and generate effective four-fermion operators. Indeed, if the large momentum k^2 flows through gluon lines then it is quite clear that the graphs in Fig. 2 can be considered as matrix elements of effective four-fermion operators.

Four-fermion operators arise also through use of equations of motion for the operators $D^\nu G_{\nu\mu}$ generated by two-loop graphs above. The sum of the two contributions is

$$T = \frac{2}{3}\frac{g^2}{k^4}\left(\sum_q\bar{q}\gamma_\mu q\right)^2 - \frac{3g^2}{k^4}\left(\sum_q\bar{q}\gamma_\mu\gamma_5 q\right)^2. \quad (23)$$

Further procedure is similar to what has been done above. Namely, we have to find the matrix elements of the four-fermion operators over virtual photons, substitute the coupling $g^2(k^2)$ by the expansion in $g^2(Q^2)$ and integrate over k^2. Then the naive expectation would be that we lose a factor of n compared to the simplest renormalon graph considered above. Indeed, since we have two renormalon chains now we are losing at least one power of log in integrals of the type (20) which in turn implies loss of a factor n. The actual result differs drastically from these expectations. Namely at large n the three-loop skeleton graphs win over the two-loop ones [17]:

$$a_n \xrightarrow{n\to\infty} = -\frac{7}{72\pi^2 b_0^2}(-b_0)^n(n+1)! \quad \text{(no anomalous dimension)}. \quad (24)$$

This dominance of the more complicated graphs at large n is due to a combination of two factors.

First, the matrix element of four-fermion operators contains an extra log compared to (18). This is easy to understand since now we have essentially the same matrix element (18) squared. Since each log in integrals of the type (20) is converted into a factor of n it is clear that we are getting an extra n. This effect is not too surprising. The other reason for the failure of the naive expectations is more profound. Let us concentrate on the expansion for the coefficient functions c_i in Eq. (15):

$$c_i(k^2) = h_0 + h_1\alpha_1(k^2) + h_2\alpha_1^2(k^2) + \cdots + h_l\alpha_1^l(k^2) + \cdots. \quad (25)$$

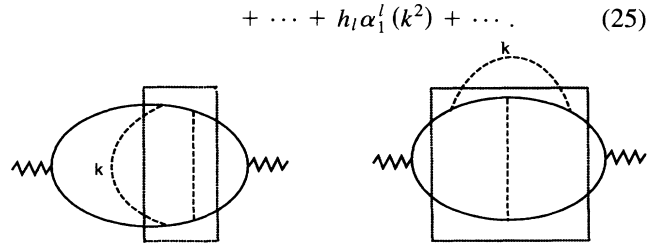

FIG. 2. Three-loop skeleton graphs giving rise to four-fermionic operators. Momentum k flowing through the gluonic lines is considered to be large so that the operator product expansion is an expansion in inverse powers of k^{-2}. The dotted boxes mark subgraphs producing four-fermionic operators.

Usually one assumes that $c_i(k)$ can be approximated by, say, the first term since $\alpha_1(k^2)$ is small. This logic does not work, however, and it is a matter of simple algebra to convince oneself that for any finite l the contribution of h_l to the asymptotic of a_n has the same large n dependence. This means, in turn, that all the terms in the expansion (25) are equally important.

Technically this happens because there are two large parameters, n and $\ln(k^2/Q^2)$, and only the logs are controlled by renormalization group. However, as a result of integration over k^2, powers of logs are converted into powers of n [see, e.g., Eq. (20)]. It turns out that lower powers of logs have larger statistical weight. As a result, the two large parameters, large combinatorial n and large log, get mixed up and the resulting contributions of all terms in the expansion (25) are of the same order. It is not ruled out, in particular, that the contributions of various h_l in Eq. (25) cancel between themselves and the true asymptotic is very different from (1).

It is worth emphasizing that this observation is specific to the ultraviolet renormalon and does not apply to the infrared renormalon (which is not summable, however, to start with). Since renormalons are reflections of the Landau poles in perturbative expansions (see, e.g., [2]) it means that the Landau pole in the ultraviolet does not necessarily manifest itself in $\Pi(Q^2)$.

To summarize, we have demonstrated that renormalon-type graphs can be evaluated by means of the operator product expansion. The construction is explicitly gauge invariant, which is especially important in case of non-Abelian gauge theories. The technique reveals, however, that the class of graphs producing one and the same asymptotic is not well defined and the size of possible Q^{-2} terms is uncertain theoretically for this reason. Moreover, properties of the sum could be very different from properties of individual terms. Assuming that this does not happen we conclude that the UV renormalon is related to effective four-fermionic interactions.

The authors are grateful to M. Beneke, V. Gribov, G. Grunberg, and M. Shifman for useful discussions. This work was supported in part by DOE under the Grant No. DE-FG02-94ER40823.

*Electronic address: vainshte@vx.cis.umn.edu

†On leave of absence from Max-Planck-Institute of Physics, Munich, Germany.

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