

Nontopological Magnetic Monopoles and New Magnetically Charged Black Holes

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The existence of nonsingular classical magnetic monopole solutions is usually understood in terms of topologically nontrivial Higgs field configurations. We show that finite energy magnetic monopole solutions also exist within a class of purely Abelian gauge theories containing charged vector mesons, even though the possibility of nontrivial topology does not even arise provided that certain relationships among the parameters of the theory are satisfied. These solutions are singular if these relationships do not hold, but even then become meaningful once the theory is coupled to gravity, for they then give rise to an interesting new class of magnetically charged black holes with hair.

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It was shown by 't Hooft and Polyakov [1] that classical finite energy magnetic monopole solutions occur in certain spontaneously broken non-Abelian gauge theories. The existence of these solutions is often understood in terms of topologically nontrivial Higgs field configurations. In this Letter we show that finite energy magnetic monopoles can also be obtained in a class of purely Abelian theories in which such topological considerations do not even arise, provided that certain relationships among the parameters of the theory are satisfied. These solutions are singular if these relationships do not hold, but even then become meaningful once the theory is coupled to gravity, for they then give rise to a new class of magnetically charged black holes with hair.

To illustrate these ideas, we consider a theory with electromagnetism coupled to a charged vector field W_μ and a neutral scalar field ϕ . The spin-1 W particles have electric charge e and a magnetic moment $gd_{\mu\nu} \equiv ieg(W_\mu^*W_\nu - W_\nu^*W_\mu)$ with g assumed to be positive [2]. They have a ϕ -dependent mass $m(\phi)$ that takes on a nonzero value $m_W = m(v)$ when the scalar field takes on its vacuum value v but vanishes at some other value of ϕ , which we arbitrarily choose to be $\phi = 0$. Adding a quartic W self-coupling proportional to $d_{\mu\nu}^2$ (other interactions are also possible), we obtain the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} |D_\mu W_\nu - D_\nu W_\mu|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{g}{4} d_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} d_{\mu\nu} d^{\mu\nu} \\ & + m^2(\phi) |W_\mu|^2 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad (1) \end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and $D_\mu W_\nu = (\partial_\mu - ieA_\mu)W_\nu$ is the U(1) covariant derivative. If $g = 2$, $\lambda = 1$, and $m(\phi) = e\phi$, this is in fact the unitary gauge form of the Lagrangian for an SU(2) gauge theory spontaneously broken to U(1) by a triplet Higgs field. Similarly, for $g = 2$, $\lambda = (\sin\theta_W)^{-2}$, and $m(\phi) = e\phi/2$ we have the unitary gauge form of the standard electroweak theory, but with all terms involving the Z or fermions omitted. However, for generic values

of g and λ an extension to a non-Abelian symmetry is not possible [3].

It is useful to display the energy density corresponding to this Lagrangian. For static configurations with $A_0 = W_0 = 0$, this may be written as

$$\begin{aligned} \mathcal{E} = & \frac{1}{4} \left(1 - \frac{g^2}{4\lambda}\right) F_{ij}^2 + \frac{g^2}{16\lambda} \left(F_{ij} - \frac{2\lambda}{g} d_{ij}\right)^2 \\ & + \frac{1}{2} |D_i W_j - D_j W_i|^2 \\ & + m^2(\phi) |W_i|^2 + \frac{1}{2} (\partial_i \phi)^2 + V(\phi). \quad (2) \end{aligned}$$

In any magnetically charged configuration, the magnetic field will be at least as singular as $1/r^2$ at the position of the magnetic charge. We will show below that in certain situations W_i can be chosen so as to cancel the singularity in the second term on the right hand side of this equation. Having done this, we are still left with a $1/r^4$ singularity from the first term. This leads to three cases: (1) If $g^2 > 4\lambda$, there are monopole configurations with negative infinite energy. The vacuum is therefore unstable against production of monopole-antimonopole pairs and the theory must be discarded. (2) If $g^2 < 4\lambda$, all monopole solutions have positive infinite energy. We will return to this case later. (3) If $g^2 = 4\lambda$, the first term on the right hand side of Eq. (2) is absent, and finite energy monopole solutions exist, as we now demonstrate.

To begin, we recall that a point magnetic charge gives rise to a radial magnetic field of magnitude Q_M/r^2 that is derived from a vector potential that necessarily possesses a Dirac string singularity along some line running from the monopole to spatial infinity. In the quantum theory, Dirac strings are acceptable as long as they cannot be detected through the Aharonov-Bohm effect by particles encircling the string. The analogous criterion in the classical theory is that the string not be detectable through the interference of waves in the W field (or any other charged field) passing on either side of the string. In both cases, this leads to the quantization condition $Q_M = q/e$ where q is either an integer or a half integer and e is the smallest electric charge in the theory [4]. We will concentrate for the present on the case $Q_M = 1/e$. This

has the advantage of allowing spherically symmetric W fields, which cannot occur [5] for any other value of Q_M , and will also allow us to make the connection with the 't Hooft–Polyakov solution.

The electromagnetic vector potential for the unit charged point monopole may be written as

$$A_i = -\epsilon_{ij3}\hat{r}_j \frac{1}{er} \left(\frac{1 - \cos\theta}{\sin^2\theta} \right). \quad (3)$$

This is spherically symmetric in the sense that the effects of a spatial rotation can be compensated by a gauge transformation. Any charged vector field that is also invariant under the same combination of rotation and gauge transformation can be written in the form

$$\begin{aligned} W_x &= -\frac{i}{\sqrt{2}} \frac{u(r)}{er} [1 - e^{i\phi} \cos\phi(1 - \cos\theta)], \\ W_y &= \frac{1}{\sqrt{2}} \frac{u(r)}{er} [1 + ie^{i\phi} \sin\phi(1 - \cos\theta)], \\ W_z &= \frac{i}{\sqrt{2}} \frac{u(r)}{er} e^{i\phi} \sin\theta. \end{aligned} \quad (4)$$

The singularities of these fields along the negative z axis are purely gauge artifacts, and can be removed by a gauge transformation that moves the Dirac string. The singularity at the origin cannot be removed by a gauge transformation but, as we shall see, it does not entail any singularity in the energy density.

The vector field (4) leads to a purely radial magnetic moment of magnitude $-|u|^2/er^2$. By setting $u(0) = \sqrt{g/2\lambda} = 1$, the $1/r^2$ singularities of F_{ij} and d_{ij} can be made to cancel in the energy density. This leaves two other potentially singular contributions. The most dangerous is the term containing the covariant curl, $D_i W_j - D_j W_i$, in which one might expect a $1/r^4$ singularity in the energy density to arise from the angular derivatives. However, explicit calculation reveals that the contributions from these angular derivatives cancel, leaving only a term proportional to $(u'/r)^2$ that causes no problem as long as $u(r) - u(0)$ is of order r^2 . The mass term could also give a singular energy density, proportional to $1/r^2$, but this can be avoided by requiring that $\phi(0) = 0$, so that $m(\phi)$ vanishes at the origin. This shielding of the magnetic charge is energetically favorable only out to a distance $R_{\text{mon}} \sim \sqrt{g} m_W^{-1}$; beyond this distance the energy is minimized if u and ϕ rapidly approach 0 and v , respectively. Standard arguments then show that configuration of minimum energy (which is a solution to the field equations everywhere except, possibly, at the origin) has a total energy $M_{\text{mon}} \sim m_W/e^2 \sqrt{g}$.

It was noted above that our Lagrangian is equivalent to an SU(2) model if $g = 2$ and $\lambda = 1$. In this case one can verify that the solution we have found is simply the familiar SU(2) monopole solution,

$$\begin{aligned} V_j^a &= \epsilon_{jak} \hat{r}_k \frac{1 - u(r)}{er}, \\ \phi^a &= h(r) \hat{r}^a, \end{aligned} \quad (5)$$

transformed into a gauge where the Higgs field has a constant direction in internal space.

Let us now return to the case $g^2 < 4\lambda$. Although the first term on the right hand side of Eq. (2) gives a divergent contribution to the energy, the remaining contributions can be made finite by the choice $u(r) = \sqrt{g/2\lambda} + O(r^2)$, $\phi(r) = O(r)$ near the origin. This leads to an energy density which at short distances is essentially that of a point monopole with a reduced charge $Q_{\text{eff}} = (1 - g^2/4\lambda)^{1/2} Q_M$. As with the previous case, u vanishes rapidly for $r > R_{\text{mon}} \sim \sqrt{g} m_W^{-1}$; in this region the energy density is simply that of an ordinary unit charged point monopole.

While infinite energy solutions such as this do not correspond to particles of the theory (1), they acquire physical significance when the theory is coupled to gravity, since the singularity can then be hidden behind the event horizon of a black hole. Let us begin by recalling the Reissner-Nordstrom solutions, which describe charged black holes in a theory governed by the coupled Einstein-Maxwell equations. For magnetic charge $Q_M = q/e$ the metric is

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (6)$$

where

$$B = A^{-1} = 1 - \frac{2MG}{r} + \frac{4\pi Gq^2}{e^2 r^2}. \quad (7)$$

while the vector potential is precisely the Dirac monopole potential of Eq. (3). Solutions with horizons exist for all values of the black hole mass greater than the extremal mass,

$$M_{\text{ext}}^{\text{RN}}(q) = \frac{\sqrt{4\pi} |q|}{e} m_{\text{pl}} \quad (8)$$

(the Planck mass $m_{\text{pl}} = G^{-1/2}$), for which the horizon radius r_H takes its minimum value,

$$r_{\text{ext}}^{\text{RN}}(q) = \frac{\sqrt{4\pi} |q|}{e} m_{\text{pl}}^{-1}. \quad (9)$$

The Reissner-Nordstrom black hole is also a solution to the spontaneously broken SU(2) theory, provided that the massive vector field vanishes and the scalar field takes on its vacuum value everywhere. However, there can also be magnetically charged black holes with “hair,” i.e., nontrivial W and ϕ fields outside the horizon. Such solutions [6] exist for a range of parameters that roughly corresponds to the condition that the “Schwarzschild radius” $2MG$ be less than R_{mon} . At large distances these approach the Reissner-Nordstrom solution, while for $r \ll R_{\text{mon}}$ the metric is approximately Schwarzschild. Thus, these solutions are most naturally viewed as 't Hooft–Polyakov monopoles with small Schwarzschild black holes located at their centers. In particular, there is no minimum value for the horizon radius, and hence no extremal solution.

The existence of these solutions can be understood by considering small fluctuations about a Reissner-Nordstrom solution with magnetic charge $1/e$. If the horizon distance $r_H \lesssim R_{\text{mon}}$, the energy density just

outside the horizon can be lowered by the creation of a W field with its magnetic moment arranged to shield the Coulomb field. One can show [7] that the Reissner-Nordstrom solution becomes unstable for r_H less than a critical value of order R_{mon} ; the configuration to which this instability leads is just such a black hole with hair.

Since these arguments do not depend on the existence of an underlying $SU(2)$ symmetry, there should also be nontrivial solutions for the more general Lagrangian of Eq. (1). Let us examine this possibility more closely. We assume static spherically symmetric matter fields as in Eqs. (3) and (4), and write the metric as in Eq. (6). The matter portion of the action then takes the form

$$S_{\text{matter}} = -4\pi \int dt dr r^2 \sqrt{AB} \left[\frac{K(u, \phi)}{A} + U_1(u, \phi) + \frac{1}{2e^2 r^4} \left(1 - \frac{g^2}{4\lambda} \right) \right], \quad (10)$$

where

$$K = \frac{(u')^2}{e^2 r^2} + \frac{1}{2} (\phi')^2, \quad (11)$$

$$U_1 = \frac{\lambda}{2e^2 r^4} \left(u^2 - \frac{g}{2\lambda} \right)^2 + \frac{u^2 m^2(\phi)}{r^2} + V(\phi). \quad (12)$$

(Primes denote differentiation with respect to r .) If we define a function $F(r)$ by

$$A(r) = \left[1 - \frac{2GF(r)}{r} + \frac{4\pi G}{r^2 e^2} \left(1 - \frac{g^2}{4\lambda} \right) \right]^{-1}, \quad (13)$$

the gravitational field equations imply that

$$F' = 4\pi r^2 \left(\frac{K}{A} + U_1 \right). \quad (14)$$

The black hole mass $M = F(\infty)$. A lower bound on this can be obtained by noting that the horizon radius r_H is a zero of $A(r)^{-1}$ and that Eq. (13) shows that such zeros can exist only if

$$F(r_H) \geq \frac{\sqrt{4\pi}}{e} m_{\text{Pl}} \left(1 - \frac{g^2}{4\lambda} \right)^{1/2} = M_{\text{ext}}^{\text{RN}} \left(1 - \frac{g^2}{4\lambda} \right)^{1/2}. \quad (15)$$

The total mass exceeds $F(r_H)$ by an amount equal to the integral of Eq. (14) from r_H to ∞ . With $u(r)$ behaving as we expect, this integral will be roughly equal to M_{mon} . Because a nontrivial $u(r)$ is energetically favorable only out to distances of order $R_{\text{mon}} \sim \sqrt{g} m_W^{-1}$, solutions of the type we seek should exist only if $r_H \leq R_{\text{mon}}$. This gives an upper bound on the mass, and implies that

$$M_{\text{ext}}^{\text{RN}} \left(1 - \frac{g^2}{4\lambda} \right)^{1/2} + M_{\text{mon}} \leq M \leq \frac{\sqrt{g} m_{\text{Pl}}^2}{2m_W} + M_{\text{mon}}, \quad (16)$$

while the horizon radius obeys

$$r_{\text{ext}}^{\text{RN}} \left(1 - \frac{g^2}{4\lambda} \right)^{1/2} \leq r_H \leq R_{\text{mon}}. \quad (17)$$

Examining the lower bounds, we see that if $g^2 \neq 4\lambda$ (i.e., if there are no finite energy monopoles) there is a new type of extremal black hole with horizon distance

and mass both less than those of the extremal Reissner-Nordstrom black hole.

Let us now consider solutions with $q \neq 1$. Except for the singular point monopole in flat spacetime or the Reissner-Nordstrom black hole, none of these can be spherically symmetric and analysis of the field equations becomes much more difficult. However, another line of attack is available. Consider first the case $g^2 < 4\lambda$, where we know that only black hole solutions are possible. The Reissner-Nordstrom solutions are classically unstable if it is energetically favorable to shield the magnetic charge by creating a cloud of W particles just outside the horizon. For $q \geq 1$ this happens if the horizon distance is less than a critical value $\sim \sqrt{gq} m_W^{-1}$ corresponding to a black hole mass $M_{\text{unstable}}(q) \sim \sqrt{gq} m_{\text{Pl}}^2 / m_W$. (The corresponding formulas for $q = \frac{1}{2}$ are obtained by replacing g with $g - 2$.) If one of these unstable solutions is perturbed, it will classically evolve to some other black hole solution. Since the total magnetic flux through the horizon must be conserved and the horizon cannot bifurcate (at least classically), this must be a solution of the type we seek, with nontrivial matter fields outside the horizon and the original magnetic charge. Thus, there must be new black hole solutions for all values of the magnetic charge such that the extremal Reissner-Nordstrom horizon distance is small enough to allow instability, i.e., for

$$1 \leq q < q_{\text{cr}} \sim e^2 g \left(\frac{m_{\text{Pl}}}{m_W} \right)^2 \quad (18)$$

and for $q = \frac{1}{2}$ if $g - 2 \geq (m_W / e m_{\text{Pl}})^2$. [If $m_W \geq e \sqrt{g} m_{\text{Pl}}$, the inequalities in Eq. (18) cannot be satisfied, and so we do not expect to find new solutions. In the context of the spontaneously broken $SU(2)$ theory, it was shown [6,8] that the static nonsingular monopole solution is absent if the vector boson mass becomes this large.] For any given charge in this range, there will be new solutions with masses ranging up to $M_{\text{unstable}}(q)$ (actually, it appears that the maximum mass is a bit higher, although of the same order of magnitude) and down to an extremal value $M_{\text{ext}}(q)$. Without spherical symmetry, we cannot derive the precise analogs of Eqs. (16) and (17). However, we expect the extremal horizon size to scale roughly with q , as it does in the Reissner-Nordstrom case, and so expect

$$M_{\text{ext}}^{\text{RN}}(q) \left(1 - \frac{g^2}{4\lambda} \right)^{1/2} + M_{\text{mon}} \leq M \leq \frac{\sqrt{gq} m_{\text{Pl}}^2}{m_W} + M_{\text{mon}}. \quad (19)$$

Matters are somewhat different if $g^2 = 4\lambda$. In this case nonsingular static solutions might be possible and could be found by minimizing the energy among a class of configurations with fixed magnetic charge. However, in the absence of spherical symmetry, the minimum energy configuration for any integral value of q might simply be a collection of infinitely separated monopoles of lower charge. This is, in fact, what apparently happens in the spontaneously broken $SU(2)$ theory (except in the Bogomol'nyi-Prasad-Sommerfield limit). The existence

of solutions with half-integer q depends on whether or not it is possible to construct a finite energy configuration with such charges (we can show that this cannot be done for $q = \frac{1}{2}$, but do not have a result for $q \geq \frac{3}{2}$). If this is possible for some half-integer values of q , then there will be a static solution with the lowest such charge.

Now consider black hole solutions when $g^2 = 4\lambda$. The Reissner-Nordstrom solutions are still unstable for small enough masses, but there is no guarantee that the end point of their classical evolution is a black hole with the same magnetic charge. For integer q , the classical instability could eventually lead to a Schwarzschild black hole plus a number of nonsingular monopoles. For half-integer q , matters are more complicated. If $g > 2$, the Reissner-Nordstrom solution with magnetic charge $1/(2e)$ is unstable for small enough mass. Since there are no nonsingular monopoles with this charge, there must be a new black hole solution with $q = \frac{1}{2}$, but there need not be any with $q > \frac{3}{2}$. If $g \leq 2$, the $q = \frac{1}{2}$ Reissner-Nordstrom solution is stable, but those with $q \geq \frac{3}{2}$ need not be. There must be either a nonsingular monopole or a new black hole solution with $q = \frac{3}{2}$, although not necessarily both; one cannot conclude anything about the solutions with higher charge.

Let us now address the formation and evolution of these new types of black holes. In theories with nonsingular monopoles of charge $Q_M = 1/e$ (i.e., those with $g^2 = 4\lambda$), a Reissner-Nordstrom black hole whose charge was an integral multiple of $1/e$ could form by the absorption of magnetic monopoles by an uncharged Schwarzschild black hole or by the collapse of matter containing magnetic monopoles. Black holes with half-integer charge could not be formed by these mechanisms. Instead, these would have to be produced in pairs, perhaps by a quantum tunneling process in a strong magnetic field. No matter what the production mechanism, evaporation via the Hawking process would cause the black hole mass to decrease and the horizon to shrink. When the mass fell below M_{unstable} , the black hole would cease to be Reissner-Nordstrom, and a W cloud would develop outside the horizon. As evaporation proceeded further and the horizon moved inward, it would be energetically favorable for nonsingular monopoles to be emitted. For a black hole with integer charge, this would continue until the charge was reduced to $1/e$; in the final stage of evaporation the horizon would disappear, leaving behind a nonsingular monopole. If there are no nonsingular monopoles with half-integer charge, black holes with half-integer charges would not evaporate completely; instead, they would eventually evolve to a black hole of minimal half-integer charge.

In theories without nonsingular monopoles (i.e., $g^2 < 4\lambda$), all magnetically charged black holes would have to be produced in pairs. The evolution of these objects would be somewhat different. As in the previous case, the Hawking process would take a Reissner-Nordstrom black

hole down to M_{unstable} , at which point a W cloud would appear outside the horizon. With further evaporation the horizon would continue to gradually contract until it had reached extremal size. Because the Hawking temperature of the resulting extremal hole vanishes, evaporation would cease at this point. However, further evolution might still be possible. Magnetic black holes with masses less than that of the extremal Reissner-Nordstrom solution have the unusual property that at large separation the Coulomb repulsion between a pair of holes is stronger than their gravitation attraction. (Similar behavior has been noted in theories with massive dilatons [9].) One could ask whether it would be possible for a black hole in this mass range to split into two holes of lower charge. This process is forbidden classically, but it might be possible quantum mechanically.

To summarize, we have shown that the existence of finite energy classical magnetic monopole solutions need not be associated with nontrivial topology. The consideration of such solutions leads naturally to a new class of magnetically charged black holes with hair.

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 - [2] If g is negative, finite energy flat spacetime monopole solutions such as we describe do not exist. However, our results concerning new black hole solutions remain true with only minor modifications.
 - [3] Apart from special cases such as those noted here, the theory described by Eq. (1) is nonrenormalizable. This will not affect the analysis of this paper, which is largely classical.
 - [4] To understand how the same quantization condition can arise in the classical and quantum theories, it is helpful to work in units where \hbar and c have not been set equal to unity. To give e the dimensions appropriate to the electric charge of a particle, the covariant derivative must be written in the form $D_\mu = \partial_\mu - i(ec/\hbar)A_\mu$. The classical quantization condition is then that ecQ_M/\hbar be an integer or half integer, which is the same as the usual quantum mechanical result.
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