A Hierarchy of Superstrings

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We construct a hierarchy of supersymmetric string theories by showing that the general N-extended superstrings may be viewed as a special class of the $(N+1)$-extended superstrings. As a side result, we find a twisted $(N+2)$ superconformal algebra realized in the N-extended string.

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Recently the remarkable discovery has been made that string theories may be interpreted as kinds of spontaneously broken phases of those with higher world-sheet symmetries [1]. In particular, it has been shown that the $N=0$ ($N=1$) strings can be viewed as a special class of vacua for the $N=1$ ($N=2$) superstrings [1–5]. It has also been shown that $N=2$ superstrings can be regarded as different phases of $N=4$ strings [6]. In this process it has been observed that a similar structure exists beyond $N=2$ [6,7], and it has been speculated that it is possible to embed the general N-extended superstrings into the $(N+1)$-extended superstrings for $N \geq 2$ [1,2,8], where by N-extended superstrings we mean those based on the linear algebras found in Ref. [9].

The purpose of this paper is to realize explicitly the N-extended superstrings as special choices of the vacua in the $(N+1)$-extended superstrings, and to give a simple proof that this $(N+1)$ formulation is equivalent to the N superstring. In fact, we will show explicitly the equivalence of the Becchi-Rouet-Stora-Tyutin (BRST) cohomology and of the operator algebra of the two formulations. As we will discuss later on, we believe that this is enough to prove the complete equivalence of the two models. Our results apply to general N superstrings for $N \geq 2$, and hence imply that there is an infinite hierarchy of superstrings. We also find a twisted $(N+2)$ superconformal symmetry realized in the N string.

We will use the N-extended superspace to describe these theories. Our conventions follow closely those of Ref. [10]: $Z=(z,\theta^i)$ with $i=1,2,\dots,N$ denoting the N-extended supercoordinates and

$$\begin{aligned} &D_i \equiv \partial_{\theta^i} + \theta^i \partial_z\,, \quad \{D_i, D_j\} = 2\delta_{ij}\partial_z\,,\\ &\theta_{12}^i \equiv \theta_1^i - \theta_2^i\,,\\ &z_{12} \equiv z_1 - z_2 - \theta_1^i\theta_2^i\,, \qquad \theta^N \equiv \frac{1}{N!}\epsilon^{j_1\cdots j_N}\theta^{j_1}\cdots\theta^{j_N}\,,\\ &\theta^{N-i} \equiv \frac{1}{(N-1)!}\epsilon^{j_1\cdots j_{N-1}i}\theta^{j_1}\cdots\theta^{j_{N-1}}\,. \end{aligned} \tag{1}$$

The N superconformal algebra (SCA) is generated by the super stress tensor T, which satisfies the operator product expansion (OPE),

$$T(Z_1)T(Z_2) \sim \frac{\theta^N}{z^2}\left(2 - \frac{N}{2}\right)T(Z_2) + \frac{\theta^{N-i}}{z}\frac{1}{2}D_iT(Z_2) + \frac{\theta^N}{z}\partial T(Z_2)\,, \tag{2}$$

where we have used the shorthand notation $z \equiv z_{12}$ and $\theta^N \equiv \theta_{12}^N$. Central extensions can appear for $N \leq 4$ [10]. However, we will mainly focus on the cases which do not have central extensions ($N \geq 5$), or have a vanishing value for the critical central charges ($N=3,4$). We will comment on the case $N=2$ later on. One can construct a string theory by using the N SCA as the (chiral) constraint algebra defining the states of the string. This information is encoded in a nilpotent BRST operator, constructed as the superspace integral of the BRST current Q_N,

$$Q_N = C_t[T + \tfrac{1}{2}T(B_t, C_t)]\,, \tag{3}$$

where the reparametrization ghosts (B_t, C_t) are fields of spin $(2 - N/2, -1)$, C_t being anticommuting for any N, and $T(B,C)$ denotes the super stress tensor of an arbitrary BC system,

$$T(B,C) = (-1)^{N(\epsilon_C+1)}\left(-\lambda_B B\,\partial C + \lambda_C\,\partial B\,C + \frac{(-1)^{\epsilon_B}}{2}D_iB\,D_iC\right). \tag{4}$$

In this last formula, λ and ϵ denote the conformal spin and Grassmann character of a field (even ϵ for bosonic fields and odd ϵ for fermionic ones), respectively, and satisfy

$$\lambda_B + \lambda_C = 1 - \frac{N}{2}\,, \qquad \epsilon_B + \epsilon_C = N\,. \tag{5}$$

We also use the following correlator for a BC system

$$C(Z_1)B(Z_2) \sim \frac{\theta_{12}^N}{z_{12}}\,. \tag{6}$$

The OPE of the BRST currents with itself can be computed and is given by

$$Q_N(Z_1)Q_N(Z_2) \sim -\frac{(-1)^N}{4}\frac{\theta^N}{z}D_i[(D_iC_t)Q_N](Z_2)\,. \tag{7}$$

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However, one can improve the BRST current by a total derivative term

$$\tilde{Q}_N = Q_N - \frac{1}{4} D_i[(D_i C_t) B_t C_t] \tag{8}$$

to make it completely nilpotent

$$\tilde{Q}_N(Z_1)\tilde{Q}_N(Z_2) \sim 0 . \tag{9}$$

This result will be helpful in constructing the general embedding to be discussed shortly and, more generally, to uncover a twisted $(N + 2)$ SCA realized in the N string.

The $(N + 1)$ SCA in N superfields is given by

$$\begin{aligned} T(Z_1)T(Z_2) &\sim \frac{\theta^N}{z^2}\left(2 - \frac{N}{2}\right)T(Z_2) + \frac{\theta^{N-i}}{z}\frac{1}{2}D_i T(Z_2) + \frac{\theta^N}{z}\partial T(Z_2), \\ T(Z_1)G(Z_2) &\sim \frac{\theta^N}{z^2}\left(\frac{3}{2} - \frac{N}{2}\right)G(Z_2) + \frac{\theta^{N-i}}{z}\frac{1}{2}D_i G(Z_2) + \frac{\theta^N}{z}\partial G(Z_2), \\ G(Z_1)G(Z_2) &\sim \frac{\theta^N}{z}2T(Z_2). \end{aligned} \tag{10}$$

This algebra can be written as in (2) using $(N + 1)$ superfields and with the $(N + 1)$ stress tensor defined in an obvious notation by $T_{N+1} \equiv G_N/2 + \theta_{N+1}T_N$. Again, all the information on physical states of the $(N + 1)$ string theory is encoded in the BRST current for the $(N + 1)$ algebra,

$$\begin{aligned} Q =& C_t T + C_g G \\ &+ C_t[\tfrac{1}{2}T(B_t, C_t) + T(B_g, C_g)] - C_g^2 B_t , \end{aligned} \tag{11}$$

where the anticommuting reparametrization ghosts C_t and the commuting supersymmetry ghost C_g have spins -1 and $-\frac{1}{2}$, respectively.

The embedding of the N string into the $(N + 1)$ string is achieved as follows. We first take an arbitrary matter background T_m for the N string which satisfies Eq. (2). We then add to this a BC system denoted by (η, ξ), with ξ anticommuting and with spin $-\frac{1}{2}$. The spin and statistics of η then follow from Eq. (5). With these ingredients at hand, we can construct the following background for the $(N + 1)$ string:

$$\begin{aligned} T &= T_m + T(\eta, \xi), \\ G &= \eta + \xi T_m + \left(\frac{N}{2} - 1\right)\xi\eta\,\partial\xi \\ &+ \frac{(-1)^N}{4}\eta(D_i\xi)^2 - \frac{1}{2}(D_i\eta)(D_i\xi)\xi , \end{aligned} \tag{12}$$

where $T(\eta, \xi)$ is the super stress tensor for the (η, ξ) system. One can show that these generators indeed satisfy the operator products given in Eq. (10).

This construction is easy to understand. The supersymmetry generator G has the structure $G \sim \eta + \tilde{Q}_N(\eta, \xi)$, where $\tilde{Q}_N(\eta, \xi)$ is the improved BRST current of Eq. (8) with the reparametrization ghosts (B_t, C_t) replaced by (η, ξ). Since the improved BRST current is fully nilpotent, only the contractions between η and $\tilde{Q}_N(\eta, \xi)$ are nonvanishing and generate the super stress tensor T, as can be imagined recalling the usual BRST algebra. Note, however, that the spin of the (η, ξ) system gets modified to the values $(\frac{3}{2} - N/2, -\frac{1}{2})$ while (B_t, C_t) had originally spin $(2 - N/2, -1)$. Actually, this construction can be extended to reveal a twisted $(N + 2)$ algebra in the N string. We will postpone the description of such an algebra to the end of the paper. Note also that one can think of (η, ξ) as fields with the same quantum number of (B_g, C_g) but opposite statistics, so that they will cancel each other through the BRST quartet mechanism.

The BRST current corresponding to the particular background just constructed is obtained by substituting Eq. (12) into Eq. (11),

$$\begin{aligned} Q_{N+1} =& C_t[T_m + T(\eta, \xi)] \\ &+ C_g\Bigg[\eta + \xi T_m + \left(\frac{N}{2} - 1\right)\xi\eta\partial\xi \\ &+ \frac{(-1)^N}{4}\eta(D_i\xi)^2 - \frac{1}{2}(D_i\eta)(D_i\xi)\xi\Bigg] \\ &+ C_t\left[\frac{1}{2}T(B_t, C_t) + T(B_g, C_g)\right] - C_g^2 B_t . \end{aligned} \tag{13}$$

To prove that this particular class of $(N + 1)$ string theories is equivalent to the N string, we perform a canonical transformation to map Q_{N+1} onto $Q_N + Q_{\text{top}}$, where Q_N was given in (3) and

$$Q_{\text{top}} = C_g \eta \tag{14}$$

gives the BRST charge for a trivial topological sector. This was the strategy already employed in Refs. [3–6], where a canonical transformation was used to map the operator algebra of the $(N + 1)$ supersymmetric formulation of the N string onto the operator algebra of the standard formulation for $N = 0, 1, 2$. In particular, we follow Ref. [4] which presented the canonical transformation factorized in three parts, each of which is of simpler construction and interpretation. The first transformation we perform is generated by

$$R_1 = \oint B_t C_g \xi , \tag{15}$$

where $\oint$ denotes superspace integration. The property of R_1 is to make T_m inert under the extra supersymmetry generated by G. We find that this transformation acts as follows:

$$Q'_{N+1} \equiv e^{R_1} Q_{N+1} e^{-R_1} = C_t[T_m + \tfrac{1}{2} T(B_t, C_t) + T(B_g, C_g) + T(\eta, \xi)] - C_g \xi T(B_g, C_g) + C_g\left(\eta + \frac{1}{2}\xi\eta\,\partial\xi + \frac{(-1)^N}{4}\eta(D_i\xi)^2\right). \quad (16)$$

Here we consistently drop total derivative terms which may appear on the right hand side of this equation, since they will not affect the BRST charge. Next, we perform a second transformation generated by

$$R_2 = \oint\left[\left(\frac{N}{4} - \frac{1}{2}\right)B_g C_g \xi\,\partial\xi + \frac{1}{4} C_g \xi\, D_i\xi\, D_i B_g + \frac{1}{8}(\xi\eta - C_g B_g)(D_i\xi)^2\right] \quad (17)$$

to simplify the BRST algebra in the (B_g, C_g, η, ξ) topological sector. This casts the BRST charge into

$$Q''_{N+1} \equiv e^{R_2} Q'_{N+1} e^{-R_2} = C_t[T_m + \tfrac{1}{2} T(B_t, C_t) + T(B_g, C_g) + T(\eta, \xi)] + C_g\eta\,. \quad (18)$$

A final transformation generated by

$$R_3 = \oint\left\{(-1)^N C_t\left[\left(\frac{3}{2} - \frac{N}{2}\right)B_g\,\partial\xi + \frac{1}{2}\partial B_g\,\xi - \frac{(-1)^N}{2} D_i B_g\, D_i\xi\right]\right\} \quad (19)$$

is used to decouple the BRST reparametrizations from the topological sector. It maps modulo total derivatives (18) onto $Q_N + Q_{\rm top}$, as given in Eqs. (3) and (14).

Obviously the BRST charges constructed out of Q_N and $Q_{\rm top}$ commute with each other and are nilpotent, giving separate conditions on the physical states. The topological charge $\oint Q_{\rm top}$ imposes the condition that the fields (B_g, C_g, η, ξ) fall into a quartet representation of the charge and decouple from the physical subspace, and we are left with the degrees of freedom of the N superstring only. Note also that our similarity transformations manifestly preserve the operator algebra. Thus the $(N + 1)$ string propagating in the background described by Eqs. (12) is equivalent to the N string.

It would be appropriate to be more careful in claiming the complete equivalence of the two formulations including scattering loop amplitudes, since our above method does not show explicitly how the integration over the moduli space and the sum over spin structures, necessary to define string amplitudes, would work out. In fact the moduli space for the N-extended super Riemann surfaces has not even been analyzed in the literature. Of course if we employ the unitarity principle, our above results are enough to establish the complete equivalence. To be more specific, one can define the N string by a Polyakov path integral, the action being given by an N-supersymmetric system coupled to N supergravity (examples of such systems can be constructed easily, e.g., in the Hamiltonian formalism). Gauge fixing to the conformal gauge (e.g., using the Lagrangian BRST formalism of Batalin and Vilkovisky) gives the type of fields considered above. Scattering amplitudes are then implicitly defined: One has to compute correlators of vertex operators corresponding to the states of the string. However, on a topologically nontrivial world sheet the choice of the conformal gauge is not strictly speaking allowed, since there are parts of the supergeometry that cannot be gauge fixed. Nevertheless, one can still gauge fix to the conformal gauge, but the action thus obtained is not completely gauge fixed since there are extra gauge symmetries present in the ghost action. In fact it is well known that on a topologically nontrivial surface there are zero modes in the ghost action [a sum of (b, c) systems with appropriate spins] which satisfy the Riemann-Roch theorem. These zero modes are responsible for the extra gauge symmetries. In particular, the gauge symmetries arising from the zero modes of the antighosts are due to the fact that we employed a too stringent gauge choice. As shown in Ref. [11], the integration over the moduli with the proper measure can be seen as arising from gauge fixing the zero modes of the antighosts. Returning to our embeddings, the fact that the extra moduli, present in the $(N + 1)$ string as compared to the N string, cancel out in the $(N + 1)$ formulation of the N string, when paired with similar "moduli" corresponding to the zero modes of the (η, ξ) system, is guaranteed by the BRST invariance of the model. The BRST symmetry organizes these moduli in a quartet with the corresponding ghost for ghosts and guarantees their decoupling, independently of the specific structure of the modular integration. One can also check that the phases to be assigned to the various spin structures of the fields in the (η, ξ) system can be chosen to assure the unitarity of the theory, as shown in Ref. [1] for the embedding $N = 0$ into $N = 1$. It is beyond the scope of the present Letter to provide an explicit proof of all these points, but it seems worthwhile to analyze them further.

So far we have restricted ourselves to the cases $N \geq 3$. For the embedding $N = 2$ into $N = 3$, one has to recall that there is a central extension $\sim c/3z^2$ appearing on the right hand side of Eq. (2) with the critical value $c = 6$. However, this central charge is balanced by a contribution $c = -6$ due to the (η, ξ) system. We have checked that our realization (12) as well as our canonical transformations also work in this case. Thus our construction is valid for arbitrary $N \geq 2$ and, when combined with the embeddings found in Ref. [1], leads to the amazing result that there exists an infinite hierarchy of superstrings.

We have also found that out of T_m and (η, ξ) one can construct an $(N + 2)$ SCA:

$$
\begin{aligned}
T(Z_1)T(Z_2) &\sim \frac{\theta^N}{z^2}\left(2-\frac{N}{2}\right)T(Z_2)+\frac{\theta^{N-i}}{z}\frac{1}{2}D_iT(Z_2)+\frac{\theta^N}{z}\partial T(Z_2),\\
T(Z_1)G^\pm(Z_2) &\sim \frac{\theta^N}{z^2}\left(\frac{3}{2}-\frac{N}{2}\right)G^\pm(Z_2)+\frac{\theta^{N-i}}{z}\frac{1}{2}D_iG^\pm(Z_2)+\frac{\theta^N}{z}\partial G^\pm(Z_2),\\
T(Z_1)J(Z_2) &\sim \frac{\theta^N}{z^2}\left(1-\frac{N}{2}\right)J(Z_2)+\frac{\theta^{N-i}}{z}\frac{1}{2}D_iJ(Z_2)+\frac{\theta^N}{z}\partial J(Z_2),\\
G^+(Z_1)G^-(Z_2) &\sim \frac{\theta^N}{z^2}\left(1-\frac{N}{2}\right)J(Z_2)+\frac{\theta^{N-i}}{z}\frac{1}{2}D_iJ(Z_2)+\frac{\theta^N}{z}\left(T+\frac{1}{2}\partial J\right)(Z_2),\\
J(Z_1)G^\pm(Z_2) &\sim \pm\frac{\theta^N}{z}G^\pm(Z_2).
\end{aligned}
\tag{20}
$$

The realization is given by

$$
\begin{aligned}
T &= T_m + T(\eta,\xi),\\
G^+ &= \eta,\\
G^- &= \xi T_m + \left(\frac{N}{2}-1\right)\xi\eta\,\partial\xi + \frac{(-1)^N}{4}\eta(D_i\xi)^2,\\
J &= \eta\xi.
\end{aligned}
\tag{21}
$$

This is the twisted $(N+2)$ SCA present in the N string. In fact, the fields (η,ξ) correspond to the ghosts (B_t,C_t), but their stress tensor is twisted in order to shift the spins from the value $(2-N/2,-1)$ to the value $(\frac{3}{2}-N/2,-\frac{1}{2})$. For $N=2$, this reduces to the special case known in the literature [12]. That such a construction was possible was suggested by Berkovits as reported in the first of Ref. [12]. One can use it to construct a consistent background for the $(N+2)$ string, and by performing canonical transformations it is possible to show that this formulation is also equivalent to the N string, as explicitly demonstrated in Ref. [6] for the case $N=2$.

Concerning linear superalgebras, it remains to be seen if one can formulate the small $N=4$ superstring [9] with SU(2) symmetry as a particular class of the large $N=4$ superstring and the general large $N=4$ superstrings depending on a free parameter x [10,13] as special backgrounds for the $N=5$ strings. Another line of investigation is to try to embed strings into W strings. For recent advances in this latter topic, see Ref. [14]. We hope to discuss these issues elsewhere.

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