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New Type of Gap Soliton in a Coupled Korteweg–de Vries Wave System

Roger Grimshaw* and Boris A. Malomed†
Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia
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We show that, in a narrow gap in the spectrum of two linearly coupled Korteweg–de Vries equations with opposite signs of the dispersion coefficient, a two-parameter family of solitons of a novel type may exist. These are envelope solitons with decaying oscillating tails, which are radically different from the gap solitons previously known in nonlinear optics. In particular, they may become singular at some value of the velocity, and degenerate into algebraic solitons in another special case. It is demonstrated that gap solitons of the same type may also exist in a nonlinear optical system consisting of focusing and defocusing tunnel-coupled planar lightguides.

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Recently, a great deal of attention has been directed to the so-called gap solitons (GS) in systems of various physical origins [1,2]. The GS exist in systems in which the spectrum of linear waves contains a forbidden gap, so that placing a soliton inside this gap may prevent it from radiative decay into linear oscillatory waves. Usually GS are sought for in systems of the nonlinear Schrödinger (NLS) type, a well-known example being the GS in a nonlinear optical system (fiber or planar waveguide) with a periodic grating [1]: two counterpropagating waves are linearly coupled by the grating-induced Bragg scattering, which gives rise to an effective dispersion in the system (even when the proper dispersion of the waveguide is neglected), necessary for supporting the soliton through competition with nonlinearity. Similarly, GS may be realized in optical systems of another type, viz., in a pair of parallel tunnel-coupled nonlinear fibers or planar waveguides. In fibers the GS exist in the temporal domain, while in the planar systems they are in the spatial domain (it is relevant to note here the experimental work of Winful, Zamir, and Feldman [1]). In this work, we aim to demonstrate a new type of GS, which can exist in coupled wave systems with *oppositely signed dispersions*. A weak linear coupling drastically changes the soliton spectrum of the system: it kills the usual solitons existing due to the nonlinearity-dispersion competition in each subsystem in the absence of the coupling, and simultaneously it opens a narrow gap in the system's linear spectrum. Solitons of a new type may exist in this gap. This general problem will be considered here in terms of a system of two linearly coupled Korteweg–de Vries (KdV) equations. It will be shown as well that essentially the same new type of GS may occur in the optical system based on a pair of tunnel-coupled focusing and defocusing planar dispersionless lightguides.

The system of coupled KdV equations is

$$u_t - \tfrac{1}{2}(u^2)_x + u_{xxx} = -\lambda v_x \,, \tag{1a}$$

$$v_t - \Delta v_x - \tfrac{1}{2}(v^2)_x - \alpha v_{xxx} = -\beta\lambda u_x \,, \tag{1b}$$

where $-\Delta$ is the relative group velocity of the linear long waves in the two subsystems, α is the relative dispersion coefficient (here we will consider only the case $\alpha > 0$, corresponding to the oppositely signed dispersions in the subsystems), λ is a small coupling constant, while β is an independent parameter [here we will consider only the case $\beta > 0$ corresponding to the situation when the system (1) is linearly stable]. The KdV equation [i.e., either (1a) or (1b) when $\lambda = 0$] is well known to describe solitary waves with exponentially decaying tails, and occurs in many physical contexts. Coupled KdV equations occur whenever the underlying physical system contains two wave modes with nearly coincident linear long-wave phase speeds [3,4]. In the context of internal gravity waves in the absence of any basic shear flow [3] the coupling between the two subsystems is through nonlinear terms and linear dispersive terms (i.e., with third deriva-

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tives) rather than the linear first order derivative terms in (1). However, coupling through linear dispersive terms will produce similar results to those described here, although we note that coupling through nonlinear terms alone presents some interesting questions (see [5]) which we shall address in future studies. In the presence of an underlying basic shear flow we can generally expect linear coupling terms with first derivatives to occur, as in (1) [4]. Before proceeding we note that the system (1) is Hamiltonian.

The spectrum for systems of KdV type is characterized by the dependence of the phase velocity c upon the wave number k. At $\lambda = 0$ the spectra of the uncoupled subsystems are $c^{(u)} = -k^2$, $c^{(v)} = -\Delta + \alpha k^2$, and they cross at the points

$$k = \pm k_0 \equiv \pm\Delta^{1/2}(1+\alpha)^{-1/2}, \quad c = c^{(0)} \equiv -\Delta(1+\alpha)^{-1}. \tag{2}$$

Note that here we must choose $\Delta > 0$. When the coupling is switched on, it prevents the crossing. One of the two generic types of the dispersion curve generated by the small coupling in the vicinity of a former crossing point is shown in Fig. 1 for the case $\alpha\beta > 0$. Elementary analysis of the dispersion relation following from the linearized form of (1) reveals that the gap shown in Fig. 1 exists, provided that $\alpha\beta > 0$, in the interval of the velocities

$$|c - c^{(0)}| < 2\sqrt{\alpha\beta}(\alpha+1)^{-1}|\lambda|. \tag{3}$$

Equality holds in (3) at the turning points of the dispersion curve where $dc/dk = 0$. In the opposite case, $\alpha\beta < 0$, the gap does not exist, and one gets the other generic dispersion curve in the vicinity of a former crossing point. However, in this case the spectrum is unstable in a range of wave numbers centered about $\pm k_0$, and the instability bubble is generated at the points where $|dc/dk| \to \infty$. In the sequel we assume that $\beta > 0$.

Only solitons with velocities belonging to the interval (3) have a chance to survive in the coupled system; all others will decay into radiation due to resonance with linear oscillatory waves. This means that the usual KdV solitons existing in the uncoupled subsystems will inevitably be destroyed, however weak the coupling (although the time of the radiative decay produced by the weak coupling may be very large [6]).

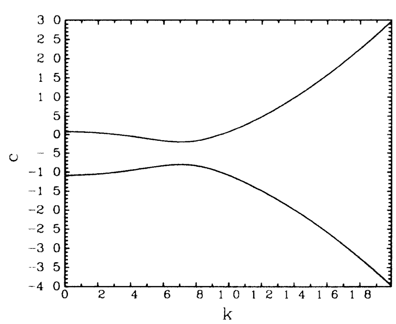

FIG. 1. The spectrum for linear waves in the presence of weak coupling in the KdV system with opposite dispersions.

To analyze the dynamics of the system inside the spectral gap, we expand the wave fields as follows:

$$u = U_1(x,t)\, e^{ik_0(x-c^{(0)}t)} + U_2(x,t)\, e^{2ik_0(x-c^{(0)}t)} + U_0(x,t) + \text{c.c.}, \tag{4a}$$

$$v = V_1(x,t)\, e^{ik_0(x-c^{(0)}t)} + V_2(x,t)\, e^{2ik_0} + V_0(x,t) + \text{c.c.}, \tag{4b}$$

where all the amplitudes U and V are assumed small and slowly varying. Substituting the expansion (4) into (1), we regard the smallness produced by differentiation of the slowly varying functions to be of the same order as the coupling constant λ. The squared amplitudes $|U_1|^2$ and $|V_1|^2$ are also assumed to be of the order of λ.

Equating coefficients in front of the second harmonics, it is straightforward to determine the corresponding amplitudes as follows,

$$U_2 = -\tfrac{1}{6}(1+\alpha)\Delta^{-1}U_1^2, \quad V_2 = \tfrac{1}{6}(1+\alpha)(\alpha\Delta)^{-1}V_1^2. \tag{5}$$

Unlike these second-harmonic amplitudes, the zeroth-harmonic amplitudes cannot be obtained explicitly at this stage [7]. Using (5), we cast the system of equations for the remaining amplitudes into the following form:

$$U_t - U_\xi + i|U|^2U - iMU = -i\epsilon V, \tag{6a}$$

$$V_t + V_\xi - i|V|^2V - iNV = -i\gamma\epsilon U, \tag{6b}$$

$$M_t + \frac{2-\alpha}{1+\alpha}M_\xi = \frac{6}{1+\alpha}(|U|^2)_\xi, \tag{7a}$$

$$N_t + \frac{1-2\alpha}{1+\alpha}N_\xi = \frac{6\alpha}{1+\alpha}(|V|^2)_\xi, \tag{7b}$$

where we use the notation

$$\xi \equiv \Delta^{-1}x + \frac{2-\alpha}{1+\alpha}t, \tag{8}$$

$$U_1 \equiv [6\Delta/(1+\alpha)k_0]^{1/2}U, \quad V_1 \equiv [6\alpha\Delta/(1+\alpha)k_0]^{1/2}V, \tag{9}$$

$M \equiv 2k_0U_0$, $N \equiv 2k_0V_0$, and $\epsilon \equiv \sqrt{\alpha}k_0\lambda$, $\gamma \equiv \beta/\alpha$. Note that to this order in the small parameter λ (or ϵ), there are no dispersive terms with second spatial derivatives in (6) and (7). Nevertheless, an effective dispersion is produced by the linear coupling terms in the system (6).

Apart from the terms involving the zeroth-harmonic amplitudes, Eqs. (6) seem formally similar to those obtained in [1] for the linearly coupled dispersionless nonlinear optical systems with a gap. The only essential difference is in the opposite signs in front of the cubic terms in (6a) and (6b), which are produced in turn through (5) by the opposite linear dispersions in (1a) and (1b). This difference proves to be crucial, radically changing the properties of the GS in the present system.

A simplified system of the same type, viz., just (6) without the terms coupling them to the zeroth-harmonic amplitudes M and N, can be obtained if one considers a pair of parallel tunnel-coupled nonlinear planar lightguides, one of which is focusing and another defocusing. In this case, t and ξ represent, respectively, the propagation coordinate and the transverse one, and the opposite signs in front of the ξ derivatives in (6a) and (6b) can be produced by different orientations of the two plane beams.

A soliton solution to (6) and (7) is now looked for in the form

$$U(\xi,t) = e^{-i\sigma t}A(\xi - wt)\,, \tag{10a}$$

$$V(\xi,t) = e^{-i\sigma t}B(\xi - wt)\,, \tag{10b}$$

$$M(\xi,t) = M(\xi - wt), \qquad N(\xi,t) = N(\xi - wt)\,, \tag{11}$$

where the frequency σ is assumed to be of order ϵ, while the velocity w may have an arbitrary order of magnitude. Insertion of (10) into (7) allows us to obtain the variables M and N as follows:

$$\begin{aligned} M &= 6[2-\alpha-(\alpha+1)w]^{-1}|A|^2, \\ N &= -6\alpha[2\alpha-1+(\alpha+1)w]^{-1}|B|^2\,. \end{aligned} \tag{12}$$

At last, substituting (10) into (6) and making use of (12) brings us to the following system of ordinary differential equations determining the possible soliton (and more general) solutions:

$$-i\sigma A - (1+w)A' - i(1+\alpha)(1+w)[2-\alpha-(1+\alpha)w]^{-1}|A|^2A = -i\epsilon B\,, \tag{13a}$$

$$-i\sigma B + (1-w)B' + i(1+\alpha)(1-w)[2\alpha-1+(\alpha+1)w]^{-1}|B|^2B = -i\gamma\epsilon A\,, \tag{13b}$$

the prime standing for differentiation in $\xi - wt$. Further analysis demonstrates that (13) have a localized solution (i.e., a soliton) for $w^2 < 1$. After some algebra, this solution can be found in the following form:

$$A = \sqrt{1-w}Re^{i\phi},\ B = \sqrt{\gamma(1+w)}Re^{i\psi}\,, \tag{14}$$

$$\begin{aligned} R^2 = |W|^{-1}(1-\Omega^2)\,[2\cosh^2(\sqrt{1-\Omega^2}z) \\ -(1-\mathrm{sgn}W\,\Omega)]^{-1}\,, \end{aligned} \tag{15a}$$

$$\begin{aligned} \tan[\tfrac{1}{2}(\phi-\psi)] = -\sqrt{(1-\Omega)/(1+\Omega)} \\ \times[\tanh(\sqrt{1-\Omega^2}z)]^{\mathrm{sgn}W}\,, \end{aligned} \tag{15b}$$

$$z \equiv \sqrt{\gamma}\epsilon(1-w^2)^{-1/2}(\xi - wt)\,, \tag{16}$$

where $\Omega \equiv [\gamma(1-w^2)]^{-1/2}\epsilon^{-1}\sigma$, and

$$W \equiv \tfrac{1}{4}(1-w^2)^{-1/2}\epsilon^{-1}\gamma^{-1/2}\,Q_1Q_2^{-1}\,, \tag{17a}$$

$$\begin{aligned} Q_1 \equiv (1+\alpha)^2(1-w^2)[(1-\gamma)(w^2-2)+(1+\gamma)w] \\ +3(2-\alpha)(1-w)^2+3\gamma\alpha(1-2\alpha)(1+w)^2\,, \end{aligned} \tag{17b}$$

$$Q_2 \equiv [2-\alpha-(1+\alpha)w][1-2\alpha-(1+\alpha)w]\,. \tag{17c}$$

We do not display here an expression for the net phase $\phi + \psi$, as it is not needed here. The structure of the soliton described by (14) to (16) is *formally* similar to the GS first obtained by Aceves and Wabnitz [1], although of course the context here is different.

In general the expression for the coefficient W may vanish whenever Q_1 vanishes. Since Q_1 (17b) is a quartic polynomial in w, there is the possibility that this could occur for four real values of w, provided of course that these all lie in the allowed range $w^2 < 1$. In the particular case $\gamma = 1$, Q_1 reduces to a cubic polynomial in w, and it can be shown that Q_1 vanishes for only one real value of w which lies in the allowed range only when $1/2 < \alpha < 2$. If also $\alpha = 1$, then this special value of w is zero. At these possible special values of w the amplitude of the soliton diverges [see (15a)], which implies that more terms should be taken into account in the expansion (4). On the other hand, the amplitude of the soliton vanishes at the two values of w for which the denominator Q_2 (17c) of the expression (17a) for W is equal to zero, i.e., at $w = (1-2\alpha)/(1+\alpha)$ and at $w = (2-\alpha)/(1+\alpha)$. These correspond to resonances with the zero-harmonic terms, and again the present theory fails.

As was mentioned above, (6) without the coupling terms to the zero-harmonic amplitudes M and N are also of interest as a possible model of coupled nonlinear focusing and defocusing planar lightguides. For this system, the soliton solution takes the same form as above with the only difference that the parameter W is now given by the expression

$$\begin{aligned} W \equiv \tfrac{1}{4}\gamma^{-1/2}\epsilon^{-1}(1-w^2)^{-1/2}[(\gamma-1)(1+w^2) \\ +2(\gamma+1)w]\,. \end{aligned} \tag{18}$$

In the general case (unless $\gamma = 1$), the expression (18) also vanishes at two special values of w.

The soliton solution contains two essential parameters: the frequency Ω and the velocity w, which is typical for envelope solitons. Indeed, coming back to the expansion (4) and taking into account that, according to (16), an effective width of the soliton scales with ϵ^{-1}, we conclude that the solution given by (14) and (15) is, actually, a typical envelope soliton with a large number of oscillations of the carrier wave fields u and v inside it. Recently, a number of studies have been devoted to analysis of envelope solitons with weakly damped oscillating tails for capillary-gravity waves [8]. These solitons are essentially NLS solitons for those turning points on the linear dispersion curve where the phase and group velocities are equal, so that the carrier and envelope have the same speeds. The solitons obtained here belong to the same class, but differ in that the linear dispersion which balances the nonlinearity derives here from the cross coupling of the two subsystems rather than from the linear second-order derivative terms in the NLS equation. They also differ in that here the velocity gap in which they can exist [see (3)] is bounded both above and below, and the gap width is proportional to the small parameter λ, whereas for the above-mentioned capillary-gravity waves, the only constraint on the soliton velocity is that it be less than (greater than) the minimum (maximum) phase velocity of the linear spectrum.

As one sees from (15), the soliton solution exists in the range of the frequencies $\Omega^2 \leq 1$. In the general case, the soliton is localized exponentially. However, its structure drastically changes in the limiting case $\Omega^2 = 1$. In the case $\Omega = \mathrm{sgn} W$ the soliton simply vanishes. However, in the limiting case $\Omega = -\mathrm{sgn} W$ it becomes an *algebraic*, i.e., weakly localized, soliton. In this limit, (15) degenerates into, for $W > 0$ and $\Omega = -1$,

$$R^2 = 2W^{-1}(1+4z^2)^{-1}, \tag{19a}$$

$$\tan[\tfrac{1}{2}(\phi - \psi)] = -2z. \tag{19b}$$

If $W < 0$, we obtain a similar limit at $\Omega = 1$:

$$R^2 = 2|W|^{-1}(1+4z^2)^{-1}, \tag{20a}$$

$$\tan[\tfrac{1}{2}(\phi - \psi)] = -(2z)^{-1}. \tag{20b}$$

The existence of this weakly localized soliton in the family of soliton solutions is a drastic difference from the previously studied type of the GS which did not have this property.

The envelope soliton has a fully stationary form (and then it has a chance to correspond to an *exact* soliton solution) if the velocity of the envelope exactly coincides with the phase velocity of the carrier wave [8]. The latter velocity may be slightly different from $c^{(0)}$ [see (2)] due to the presence of the small frequency σ in (10), and of a small additional wave number δk, which according to (10), may be defined as the limit value of ϕ' (or ψ') at infinity. It is easy to find directly from (10) that $\delta k = w\sigma(1-w^2)^{-1}$. Finally, with regard to the fact that σ is small, we find the following corrected value for the phase velocity of the carrier wave:

$$\begin{aligned} c &= (c^{(0)}k_0 + \sigma)/(k_0 + \delta k) \\ &\approx c^{(0)} + (\sigma/k_0)[1 - c^{(0)}w(1-w^2)^{-1}]. \end{aligned} \tag{21}$$

As concerns the envelope's full velocity, it follows from (16) and (8) that in the original variables (x,t), it is $[w - (2-\alpha)(1+\alpha)^{-1}]\Delta$. This expression should be equated to (21) to find the value of w at which we have the fully stationary soliton. At $\sigma = 0$ this happens at

$$w = w_* \equiv \frac{1-\alpha}{1+\alpha}. \tag{22}$$

A nonzero value of σ will slightly shift the value (22).

Inserting the value (22) into the expression (17) or (18), we notice that the coefficient W *does not* generally vanish at this value of the velocity; i.e., in the general case we indeed get a well-defined solution for the stationary soliton. However, in the case $\alpha = \gamma = 1$, when the two KdV subsystems are fully symmetric with respect to each other, the value $w = w_* = 0$ [from (22)] gives $W = 0$. So, in this symmetric case the shape of the stationary soliton remains unknown. A nonzero value of σ would make W finite, but on substituting this into (15a), we obtain the soliton's amplitude in which the small parameter ϵ is nullified by a small value of σ, so that we have a finite amplitude for which, strictly speaking, the expansion (4) is not legitimate. Nevertheless, it is natural to expect that the soliton's amplitude should be anomalously large in this fully symmetric case. To clarify this issue, numerical simulations are necessary and we are currently investigating this.

It might be expected that the soliton whose envelope is moving at a velocity different from that of the carrier wave will slowly change its velocity due to a weak emission of radiation (this process is beyond the framework of the asymptotic procedure employed in this work), and will eventually attain the value of the velocity corresponding to the stationary soliton. A full consideration of the nonstationary behavior of the envelope soliton also requires further numerical simulations.

In conclusion, we have found a new type of gap soliton which we believe is generic. Although our analysis here is based on the coupled KdV system (1), it seems clear that the essential ingredient is the spectral gap shown in Fig. 1, and it is well known that this is one of the two generic possibilities when two subsystems have coincident phase speeds when uncoupled, and are then coupled through a linear mechanism. In this context we mention for instance that there have been found recently some particular models of periodic mechanical media which have such a narrow gap in the spectrum of linear *acoustic* waves [9]. In this case, our results imply the possibility of the existence of an acoustic gap soliton.

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* Author to whom corresondence should be addressed.
† Permanent address: Department of Applied Mathematics, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel. Electronic address: malomed@leo.math.tau.ac.il

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