Self-Dual Gravity and the Chiral Model

Viqar Husain
Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2J1
(Received 28 September 1993)

The self-dual Einstein equation (SDE) is shown to be equivalent to the two dimensional chiral model, with gauge group chosen as the group of area preserving diffeomorphisms of a two dimensional surface. The approach given here leads to an analog of the Plebanski equations for general self-dual metrics, and to a natural Hamiltonian formulation of the SDE, namely that of the chiral model.

PACS numbers: 04.60.Ds, 11.10.Lm, 11.30.Rd

Since the pioneering works on the Korteweg–de Vries (KdV) equation, two dimensional integrable models have been studied extensively and there a number of books that discuss the developments [1–3].

There are some basic features shared by all the models which provides the clues to integrability. One of these is the existence of two different Hamiltonian formulations for the same equation. This feature allows a systematic way to construct the conserved quantities and to prove that these are in involution. Another is the existence of the linear Lax form of the nonlinear equations. This is basically the same as the two dimensional zero curvature conditions that lead to the nonlinear equations. An example of this is the derivation of the KdV equation from an SL(2,R) zero curvature condition using a particular parametrization of the gauge field [2].

More recently it has been demonstrated that many integrable equations are also derivable from Yang-Mills self-duality conditions in four dimensions. For example, using again various parametrizations of an SL(2,R) gauge field, and this time imposing self-duality on the curvatures followed by a dimensional reduction to two dimensions, it has been shown that one can obtain the KdV, sine-Gordon, and nonlinear Schrodinger equations [4–6]. It has also been shown that one dimensional reductions of the Yang-Mills self-duality conditions lead to various known classical equations depending on the choice of gauge group [7].

The self-dual Einstein equation (SDE) is another system studied extensively. Plebanski [8] has given an elegant formulation of these equations in terms of one function of all the spacetime coordinates, referred to as the heavenly equations. There are indications that this system is entirely integrable [9] although it has not yet been shown to have the standard features associated with integrability that are mentioned above. There are a number of interesting results associated with these equations. Using a form of the SDE [10] suggested by the Ashtekar Hamiltonian variables for general relativity [11], a connection with the self-dual Yang-Mills equation has been demonstrated [12]: the SDE may be obtained from a 0+1 dimensional reduction of the self-dual Yang-Mills equation when the gauge group is chosen as the group of volume preserving diffeomorphisms of an (auxilliary) three manifold. Another result is that the field equation for the continuum limit of the Toda model is the same as the SDE for a special ansatz for the metric [13]. A further connection with two dimensional theories has been the derivation of the Plebanski equation [8] for self-dual metrics from a large N limit of the SU(N) chiral model [14].

In this Letter it is shown how the chiral model field equations may be derived in a simple and direct way from the (unreduced) SDE. The starting point will be the relatively new way of writing the SDE in a 3+1 form due to Ashtekar, Jacobson, and Smolin [10].

The chiral field $g(x,t)$ is a mapping from a 2D spacetime into a group g. The dynamics follows from the Lagrangian density

$$L = \tfrac{1}{2}\mathrm{Tr}(\partial_\mu g^{-1}\partial_\nu g)\eta^{\mu\nu}, \tag{1}$$

where $\eta^{\mu\nu}$ is the flat Minkowski or Euclidean metric. The equations of motion are

$$\partial_\mu(g^{-1}\partial_\mu g) = 0. \tag{2}$$

If we define the Lie algebra valued 1-form $A_\mu := g^{-1}\partial_\mu g$, then this equation of motion becomes

$$\partial_\mu A_\mu = 0. \tag{3}$$

Since A_μ by definition has a pure gauge form, it follows that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{4}$$

Associated with the gauge field A_μ there is also the covariant derivative

$$D_\mu = \partial_\mu + A_\mu. \tag{5}$$

The chiral model describes flat connections A_μ satisfying $\partial_\mu A_\mu = 0$. Equations (3) and (4) are the first order forms of the field equation (2).

The SDE can also be written in a first order form using the Ashtekar Hamiltonian variables for general relativity [10]. Self-duality is the essential ingredient for this canonical formulation and it is natural to ask how the SDE looks in it. The phase space coordinate is the spatial projection of the (anti)self-dual part of the spin connection and its conjugate momentum is a densitized

0031-9007/94/72(6)/800(4)$06.00

dreibein. The same is true for Euclidean or (2,2) signatures, or complex general relativity, which are the cases of interest for self-dual Riemann curvatures.

In these Hamiltonian variables, we would like to know what is the phase space condition corresponding to the vanishing of the (anti)self-dual part of the four dimensional Riemann curvature. The answer is that the spatial projection of the latter must be zero. The vanishing of this spatial projection, when substituted into Ashtekar's 3+1 evolution equations leads to the new form of the SDE. It is straightforward to verify that this condition remains zero under the Hamiltonian evolution. The resulting equations on four-manifolds $M = \Sigma^3 \times R$ may be written in terms of three spatial vector fields V_i^a on Σ^3:

$$\mathrm{Div} V_i^a = 0, \tag{6}$$

$$\frac{\partial V_i^a}{\partial t} = \frac{1}{2}\epsilon_{ijk}[V_j, V_k]^a, \tag{7}$$

where the divergence is defined with respect to a constant auxiliary density and the right hand side of (7) is the Lie bracket. The self-dual four metrics are constructed from solutions of these equations using

$$g^{ab} = (\det V)^{-1}[V_i^a V_j^b \delta^{ij} + V_0^a V_0^b]. \tag{8}$$

Here $i, j, k = 1,2,3$ label the vector field, $a, b, ...$ are abstract vector indices, V_0^a is the vector field that is used to perform the 3+1 decomposition, and $\partial V_i^a/\partial t \equiv V_0^b \partial_b V_i^a$. The time derivative in (7) can be written in the more general form $[V_0, V_i]^a$. (For details of the derivation of these equations the reader is referred to [10] where they were originally derived, or the review in [15].)

The starting point will be the SDE in the form (6) and (7). We first rewrite Eq. (7) in a form similar to that suggested by Yang [16] for the self-dual Yang-Mills equation

$$F_{ab} = \tfrac{1}{2}\epsilon_{ab}{}^{cd}F_{cd} \tag{9}$$

on a complex manifold. Replacing the (local) complex flat coordinates $x_0, ..., x_3$ by the linear combinations $t = x_0 + ix_1$, $u = x_0 - ix_1$, $x = x_2 - ix_3$, and $v = x_2 + ix_3$, Eq. (9) becomes

$$F_{tx} = F_{uv} = 0, \tag{10}$$

$$F_{tu} + F_{xv} = 0. \tag{11}$$

For the SDE, defining in a similar way

$$\mathcal{T} = V_0 + iV_1, \qquad \mathcal{U} = V_0 - iV_1,$$
$$\mathcal{X} = V_2 - iV_3, \qquad \mathcal{V} = V_2 + iV_3, \tag{12}$$

the evolution equations (7) become

$$[\mathcal{T}, \mathcal{X}] = [\mathcal{U}, \mathcal{V}] = 0, \tag{13}$$

$$[\mathcal{T}, \mathcal{U}] + [\mathcal{X}, \mathcal{V}] = 0, \tag{14}$$

where the vector indices have been suppressed. This shows a rather direct analogy between the self-dual Yang-Mills and Einstein equations, namely, the Yang-Mills curvatures in Eqs. (10) and (11) are replaced by the Lie brackets of the vector fields. [This analogy has been noted in a related way in Ref. [12], and Eqs. (13) and (14) have also been studied in Ref. [17] from a different viewpoint than that below.]

We now show how the SDE may be written as a chiral model field equation. Fixing a local coordinate system t, x, p, q, the volume form is $\omega = dt \wedge dx \wedge dp \wedge dq$, with respect to which we define the divergence in Eq. (6). We take now the following divergence free form for the vector fields $\mathcal{T}, \mathcal{X}, \mathcal{U}, \mathcal{V}$ in terms of two functions, $A_0(t,x,p,q)$ and $A_1(t,x,p,q)$:

$$\mathcal{T}^a = \left(\frac{\partial}{\partial t}\right)^a, \quad \mathcal{X}^a = \left(\frac{\partial}{\partial x}\right)^a,$$
$$\mathcal{U}^a = \left(\frac{\partial}{\partial t}\right)^a + \alpha^{ba}\partial_b A_0, \quad \mathcal{V}^a = \left(\frac{\partial}{\partial x}\right)^a + \alpha^{ba}\partial_b A_1, \tag{15}$$

where $\alpha^{ab} = (\partial/\partial p)^{[a} \otimes (\partial/\partial q)^{b]}$ is the antisymmetric tensor that is the inverse of the two form $(dp \wedge dq)_{ab}$ in the (p,q) plane. (This form for the vector fields is similar to but more general than that used previously by the author in Ref. [15], where one-Killing-field reductions of the self-duality equations are discussed.) Substituting Eqs. (15) into (13) and (14) gives

$$\alpha^{ab}\partial_b\left[\partial_0 A_1 - \partial_1 A_0 + \{A_0, A_1\}\right] = 0, \tag{16}$$

$$\alpha^{ab}\partial_b\left[\partial_0 A_0 + \partial_1 A_1\right] = 0, \tag{17}$$

where the bracket on the left hand side of Eq. (16) is the Poisson bracket with respect to α^{ab}:

$$\{A_0, A_1\} := \alpha^{ab}\partial_a A_0 \partial_b A_1 = \partial_p A_0 \partial_q A_1 - \partial_q A_0 \partial_p A_1, \tag{18}$$

and ∂_0, ∂_1 denote partial derivatives with respect to t, x, etc. Equations (16) and (17) imply that the terms in their square brackets are equal to two arbitrary functions of t, x, which we write as

$$\partial_0 A_1 - \partial_1 A_0 + \{A_0, A_1\} = \partial_0 F(x,t) + \partial_1 G(x,t), \tag{19}$$

$$\partial_0 A_0 + \partial_1 A_1 = \partial_1 F(x,t) - \partial_0 G(x,t) \tag{20}$$

[where $F(x,t), G(x,t)$ are arbitrary]. With the redefinitions

$$a_0(t,x,p,q) := A_0 + G, \qquad a_1(t,x,p,q) := A_1 - F, \tag{21}$$

Eqs. (19) and (20) become

$$\partial_0 a_1 - \partial_1 a_0 + \{a_0, a_1\} = 0, \tag{22}$$

$$\partial_0 a_0 + \partial_1 a_1 = 0. \tag{23}$$

These are precisely the chiral model Eqs. (3) and (4) on the x, t "spacetime," with p, q treated as coordinates on an "internal" space, and with the commutator in (4) replaced by the Poisson bracket with respect to α^{ab}. The gauge group is therefore the group of transformations that preserve α^{ab} on the internal p, q space. [Note that the redefinitions (21) do not alter the vector fields $\mathcal{U}, \mathcal{V}$ in Eqs. (15).]

There is an important question regarding the relation between the full SDE (7), and the chiral model equations (22) and (23) derived from them. How general is the form (15) for the vector fields? We can see that the two first order equations for a_0, a_1 are equivalent to a single second order equation for a function $\Lambda(t, x, p, q)$. Equation (23) implies

$$a_0 = \partial_1 \Lambda, \qquad a_1 = -\partial_0 \Lambda. \tag{24}$$

With this the vector fields $\mathcal{U}, \mathcal{V}$ in Eq. (15) become

$$\begin{aligned} \mathcal{U} &= \frac{\partial}{\partial t} - \Lambda_{xq}\frac{\partial}{\partial p} + \Lambda_{xp}\frac{\partial}{\partial q}, \\ \mathcal{V} &= \frac{\partial}{\partial x} + \Lambda_{tq}\frac{\partial}{\partial p} - \Lambda_{tp}\frac{\partial}{\partial q}, \end{aligned} \tag{25}$$

and Eq. (22) becomes

$$\Lambda_{tt} + \Lambda_{xx} + \Lambda_{xp}\Lambda_{tq} - \Lambda_{xq}\Lambda_{tp} = 0 \tag{26}$$

(where the subscripts denote partial derivatives). Equation (26) is one equation for a function of all the spacetime coordinates and therefore does not represent any reduction in the local degrees of freedom for self-dual metrics. Using Eq. (8) the line element is

$$\begin{aligned} ds^2 = \; & dt(\Lambda_{tp}dp + \Lambda_{tq}dq) + dx(\Lambda_{xp}dp + \Lambda_{xq}dq) \\ & - \frac{1}{\{\Lambda_t, \Lambda_x\}}\big[(\Lambda_{xp}dp + \Lambda_{xq}dq)^2 \\ & + (\Lambda_{tp}dp + \Lambda_{tq}dq)^2\big]. \end{aligned} \tag{27}$$

For comparison, and to see the generality of the form of the vector fields used in Eq. (25), we note how Plebanski's first heavenly equation for general self-dual metrics may be derived from Eqs. (13) and (14) [17]. Working again in specific coordinates and taking the same form for $\mathcal{T}, \mathcal{X}$ as in Eq. (15) [which solves the first of Eqs. (13)], let, for some function $\Omega(t, x, p, q)$,

$$\begin{aligned} \mathcal{U} &= -\Omega_{xq}\frac{\partial}{\partial p} + \Omega_{xp}\frac{\partial}{\partial q}, \\ \mathcal{V} &= \Omega_{tq}\frac{\partial}{\partial p} - \Omega_{tp}\frac{\partial}{\partial q}. \end{aligned} \tag{28}$$

The vector fields (28) solve Eq. (14), while the second equation in (13) leads (after a few steps) to Plebanski's first equation

$$\Omega_{xp}\Omega_{tq} - \Omega_{xq}\Omega_{tp} = 1. \tag{29}$$

A comparison of Eqs. (25) and (28) shows the vector fields $\mathcal{U}, \mathcal{V}$ in each equation have the same functional content. Therefore Eq. (26) is an alternative to Plebanski's Eq. (29).

An advantage of Eq. (26) over the Plebanski one (29) is that the former has a natural Hamiltonian formulation which is just that of the chiral model. This Hamiltonian formulation is given in, for example, Ref. [1] for finite dimensional groups, and its generalization to the infinite dimensional case of relevance here is immediate.

There is now also the possibility of approaching the SDE directly from the two dimensional model point of view, and investigating integrability using the standard methods. For example, one can derive conservation laws [18] via this approach, and ask if there is a second Hamiltonian formulation just as for other integrable models.

However some remarks are in order regarding this because global spacetime considerations need to be addressed before a Hamiltonian can be written down. The internal gauge group for the chiral model must first be fixed to be the group of area preserving diffeomorphisms of a *specific* two dimensional surface. This fixes part of the topology of the self-dual manifold. The topology of the two dimensional chiral model background remains to be specified. From this viewpoint therefore, there is not one but an infinite number of Hamiltonian formulations specified by the gauge group and the chiral model background, with each phase space associated with a particular sector of self-dual metrics. A further question in this regard is how large a class of solutions to the SDE results from a given internal group and chiral model background. Whereas the chiral model solution space is infinite dimensional, the metrics derived from the solutions may be related by diffeomorphisms since the coordinates have been only partially fixed in (15). In summary, care is needed in making statements of a global nature given that all derivations in this paper involve local considerations.

Investigating the quantum theory via canonical quantization may also be of interest since the SDE constitute the largest midi-superspace model. The existence of an infinite number of conservation laws for this system [18–20], unlike the full Einstein equations [21], leads to the possibility of an infinite number of fully gauge invariant classical observables to represent as linear operators on the Hilbert space of the theory. A proper quantization should lead to a description of quantum "nonlinear gravitons" [9].

In summary, we have shown how the chiral model field equations can represent the full SDE starting from the Ashtekar-Jacobson-Smolin form of the latter. This result also gives an alternative to the Plebanski equations.

This work was supported by NSERC of Canada.

[1] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin,

1987).
[2] A. Das, *Integrable Models* (World Scientific, Singapore, 1989).
[3] A. C. Newell, *Solitons in Mathematical Physics* (SIAM, Philadelphia, 1985).
[4] R. S. Ward, Nucl. Phys. **B236**, 381 (1984).
[5] L. J. Mason and G. A. J. Sparling, Phys. Lett. A **137**, 29 (1989).
[6] I. Bakas and D. A. Depireux, Int. J. Mod. Phys. A **7**, 1767 (1992).
[7] S. Chakravarty, P. A. Clarkson, and M. J. Ablowitz, Phys. Rev. Lett. **65**, 1085 (1990).
[8] J. Plebanski, J. Math. Phys. **16**, 2395 (1975).
[9] R. Penrose, Gen. Relativity Gravitation **7**, 31 (1976).
[10] A. Ashtekar, T Jacobson, and L. Smolin, Commun. Math. Phys. **115**, 631 (1988).
[11] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987).
[12] L. Mason and E. T. Newman, Commun. Math. Phys. **121**, 659, (1989).
[13] I. Bakas and E. Kiritsis, in *Topological Methods in Field Theory*, edited by W. Nahm, S. Randjbar-Daemi, E. Sezgin, and E. Witten (World Scientific, Singapore, 1991).
[14] Q. H. Park, Phys. Lett. B **236**, 429 (1990); Phys. Lett. B **238**, 287 (1990).
[15] V. Husain, Classical Quantum Gravity **10**, 543 (1993).
[16] C. N. Yang, Phys. Rev. Lett. **38**, 1377 (1977).
[17] S. Chakravarty, L. Mason, and E. T. Newman, J. Math. Phys. **32**, 1458 (1991).
[18] V. Husain, Report No. Alberta-Thy-47-93 [Classical Quantum Gravity (to be published)].
[19] T. A. B. Strachan, Classical Quantum Gravity **10**, 1417 (1993).
[20] J. D. E. Grant, Report No. DAMPT-R92/47 (to be published).
[21] C. G. Torre and I. M. Anderson, Phys. Rev. Lett. **70**, 3525 (1993).