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Bell's Inequalities versus Teleportation: What is Nonlocality?

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The nonlocality responsible for violations of Bell's inequalities is not equivalent to that used in teleportation, although they probably are two aspects of the same physical property. There are mixed states which do not violate any Bell type inequality, but still can be used for teleportation.

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In his pioneering work, Bell [1] proved that in general two quantum systems cannot be considered as separate even if they are located far from each other. When measurements are performed independently on each of the sytems, their results are correlated in a way which cannot be explained by any local model. Bell considered some particular situations—two spin-$\frac{1}{2}$ particles in a singlet state and equivalent cases—but later works [2] have shown that nonseparability is generic. Any pure entangled (nonfactorizable) state of any number of systems generates nonlocal correlations.

Recently another aspect of quantum nonseparability has been discovered. Bennett *et al.* [3] have shown that two spin-$\frac{1}{2}$ particles, separated in space and entangled in a singlet state, can be used for teleportation. By teleportation they mean the following. Suppose that one observer, Alice, is given a spin-$\frac{1}{2}$ particle in a state Φ unknown to her, and her task is to help a friend, Bob, situated far away, prepare another spin-$\frac{1}{2}$ particle in the same state Φ. But Alice is not allowed to send to Bob the particle she was given, nor another particle which interacted with it. Alice could perform some measurements on the particle she is given and communicate the result to Bob, but since she is given a single particle she cannot learn much about its state Φ from the results of her measurement. On the other hand, suppose that Alice and Bob also share a pair of spin-$\frac{1}{2}$ particles in a singlet state. Then Alice can perform a combined measurement on the particle she was given *and* on her singlet particle, and communicate the result to Bob. Depending on the message he receives, Bob performs a certain rotation on his singlet particle and brings it into the state Φ. Roughly speaking, Bob's success resides in the fact that actually two different kinds of information have been sent to him. He received classical information—Alice communicated to him the result of her measurement—and nonlocal information, transmitted somehow through the nonseparability of the singlet pair.

These two aspects of nonseparability, the Bell correlations and the capacity for teleportation, seem to be equivalent. When the two particles constituting the nonlocal channel are maximally entangled [4] (as in the singlet state), the unknown state Φ can be teleported from Alice to Bob with perfect accuracy. (In practice, one should of course consider the effects of noise, and of different errors affecting the measurements.) On the other hand, as Bennett *et al.* noted, states which are less entangled still could be used for teleportation but they "reduce the fidelity of teleportation and/or the range of states Φ which can be accurately teleported." Finally, if the two particles are in a direct product state they do not violate any Bell inequality and are also useless for teleportation. It seems thus very natural to make the following conjecture.

Conjecture.— Whenever two systems which are separated in space violate some Bell inequality, they can be used for teleportation, and vice versa.

In the above conjecture by "can be used for teleportation" we mean that by using these two systems Alice can transmit to Bob the unknown state Φ better, according to some appropriate scale, than by using only a classical channel.

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Surprisingly enough, the above conjecture is wrong. In the present Letter I will show that there are situations in which two systems separated in space do not violate any Bell inequalities but still are useful for teleportation. The above situation arises when the two systems are not in a pure state but in a mixture.

From the point of view of Bell inequalities, mixtures have a peculiar behavior. Any pure entangled state violates Bell's inequalities, but mixtures of entangled states do not necessarily violate them. Consider, for example, a mixture of the singlet state

$$|\Psi^-\rangle=(1/\sqrt{2})(\uparrow\downarrow-\downarrow\uparrow)\,, \tag{1}$$

and the triplet state

$$|\Psi^+\rangle=(1/\sqrt{2})(\uparrow\downarrow+\downarrow\uparrow)\,, \tag{2}$$

both of them with equal weights. Both $|\Psi^-\rangle$ and $|\Psi^+\rangle$ are entangled—they are even maximally entangled—but this mixture is completely equivalent, with respect to any measurements, to a mixture of direct products

$$|\Psi_1\rangle=\uparrow\downarrow \tag{3}$$

and

$$|\Psi_2\rangle=\downarrow\uparrow \tag{4}$$

with equal weights. Since both $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are direct products, none of them violates Bell's inequalities, and thus the mixture does not violate them either. Obviously, such a mixture cannot be used for teleportation.

It is clear that any mixture which is equivalent to a mixture of direct products does not violate Bell's inequalities. But direct products are the only pure states which do not violate Bell's inequalities, so one might think that if a mixture *is not* equivalent to a mixture of direct products, it must violate some Bell inequality. Surprisingly this is not the case. Werner, in a paper [5] too rarely cited, showed that there are mixtures which are not equivalent to mixtures of direct products but which do not violate any Bell inequality. All the correlations obtained with such mixtures could be obtained from a local hidden variable model. A particular case of such a mixture is that of two spin-$\frac{1}{2}$ particles described by the density matrix

$$W=\tfrac{1}{8}I+\tfrac{1}{2}|\Psi^-\rangle\langle\Psi^-|\,, \tag{5}$$

where I is the identity operator and $|\Psi^-\rangle$ is the singlet state (1). When measurements of spin polarization are performed, say in the $\hat{\xi}$ direction for the first particle and in the $\hat{\eta}$ direction for the second, the probability of obtaining the result "up,up" is

$$P(S^1_{\hat{\xi}}=+1,S^2_{\hat{\eta}}=+1)=\tfrac{1}{4}(1-\tfrac{1}{2}\cos\alpha)\,, \tag{6}$$

where α is the angle between the directions $\hat{\xi}$ and $\hat{\eta}$.

The same joint probabilities are given by the following probabilistic local hidden variable model. Let the hidden local variable be a vector

$$\hat{\boldsymbol{\lambda}}=\sin\theta\cos\phi\hat{\mathbf{i}}+\sin\theta\sin\phi\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}}\,, \tag{7}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors along the x, y, and z axes. Both particles in each pair are given the same $\hat{\boldsymbol{\lambda}}$, and different pairs in the ensemble have different $\hat{\boldsymbol{\lambda}}$'s, uniformly distributed on the unit sphere. That is, the distribution function of the hidden variable $\hat{\boldsymbol{\lambda}}$ is

$$d\rho(\hat{\boldsymbol{\lambda}})=\frac{1}{4\pi}\sin\theta d\theta d\phi\,. \tag{8}$$

When spin polarization measurements are performed on each particle in a pair, each particle yields the outcomes "up" or "down" according to a local scheme, without any need to "communicate" with its partner. The schemes are as follows. Particle 1 gives the answer "up" with probability

$$P(S^1_{\hat{\xi}}=+1,\hat{\boldsymbol{\lambda}})=\cos^2(\alpha_1/2)\,, \tag{9}$$

where α_1 is the angle between the directions $\hat{\xi}$ and $\hat{\boldsymbol{\lambda}}$. Particle 2 yields "up" with probability

$$P(S^2_{\hat{\eta}}=+1,\hat{\boldsymbol{\lambda}})=\begin{cases}1 & \text{if } 2\cos^2(\alpha_2/2)<1\,,\\ 0 & \text{if } 2\cos^2(\alpha_2/2)>1\,,\end{cases} \tag{10}$$

where α_2 is the angle between $\hat{\eta}$ and $\hat{\boldsymbol{\lambda}}$. It is easy to verify that the joint probabilities given by this local hidden variables model,

$$P_{\rm LHV}(S^1_{\hat{\xi}}=+1,S^2_{\hat{\eta}}=+1)=\int d\rho(\hat{\boldsymbol{\lambda}})P(S^1_{\hat{\xi}}=+1,\hat{\boldsymbol{\lambda}})P(S^2_{\hat{\eta}}=+1,\hat{\boldsymbol{\lambda}})\,, \tag{11}$$

are identical to those given by quantum mechanics, Eq. (6).

Mixtures of this type might raise an uneasy feeling. On one hand one feels intuitively that they are nonlocal, as they cannot be obtained as mixtures of local states (direct products). On the other hand, this supposed nonlocality does not appear in the correlations—all correlations generated by such a mixture are classical. As I will show now, the nonseparability of these mixtures is revealed by teleportation. Using particles in the mixed state (5) one can teleport an unknown state Φ, albeit not with 100% fidelity, but better than by using only a classical communication channel.

Consider the following situation. Alice is given a spin-$\frac{1}{2}$ particle in a pure state Φ, unknown to her, and Bob is required to prepare a spin-$\frac{1}{2}$ particle in a pure or a mixed state, as close as possible to Φ. In general Bob will not succeed in perfectly reproducing Φ, but will prepare some other state, denoted by Φ' if it is pure, or by ρ if it is mixed. Let the measure of success—the "score" obtained by Bob—be $|\langle\Phi|\Phi'\rangle|^2$ or, equivalently $\mathrm{Tr}(\rho|\Phi\rangle\langle\Phi|)$. Suppose now that this "quantum game" is repeated many times. Each time Alice gets a particle polarized in a different direction, uniformly distributed on the unit sphere. Suppose also that Alice knows what the distribution of directions is, but, of course, not the direction of polarization of the individual particles. The final score of the game—the average of the scores obtained in the different runs—is a measure of the fidelity of the transmission. (Of course many other definitions of fidelity could be given and each of them could be more appropriate for studying different aspects of the transmission. But the present Letter is conceived as a counter example: It is sufficient to find a particular aspect in which teleportation works and Bell's inequalities do not, to establish the inequivalence of these two types of nonlocality.)

Some typical scores are as follows. If Alice and Bob cannot communicate at all, Bob could try to guess the state, according to some particular scheme. (Bob is always required to prepare a particle in some state.) The results of any such guess scheme will always [6] be $\frac{1}{2}$. On the other hand, if in any run, Alice and Bob are connected by both a classical and a nonlocal communication channel, the latter being formed by two spin-$\frac{1}{2}$ particles in a singlet state, they can perform a standard teleportation scheme. In this case the success is total, and the fidelity is 1, the maximal possible. Finally, if Alice and Bob are connected only by a classical channel, the best possible score is [6] $\frac{2}{3}$. A particular way to obtain this score is the following. Each run, Alice measures the spin along the z direction, and tells the result to Bob. Bob prepares a particle polarized along the z axis, up or down, according to the result of Alice's measurement. But suppose now that Alice and Bob share pairs of spin-$\frac{1}{2}$ particles in the mixed state (5). Then they can perform a standard teleportation scheme and get the score of $\frac{3}{4}$, better than what could be obtained classically. Indeed, the mixture (5) can be viewed as a mixture of 50% the completely undetermined mixed state $W=\frac{1}{4}I$ and 50% the pure state $|\Psi^-\rangle$. When the state is completely undetermined, the result of the teleportation is completely random, so the score of the quantum game is $\frac{1}{2}$, while when the state is the singlet, the teleportation is absolutely successful, and the score is 1. Therefore the total score obtained by using the teleportation procedure with the mixture (5) is the average of the above two scores, that is

$$\text{fidelity}=\tfrac{1}{2}\times\tfrac{1}{2}+\tfrac{1}{2}\times 1=\tfrac{3}{4}\,. \qquad (12)$$

In conclusion, the nonlocality revealed by teleportation is not equivalent to the nonlocality of quantum correlations, although they are probably two faces of the same physical property. Needless to say, the experimental verification of teleportation without violation of Bell's inequalities is more than desirable.

Many interesting questions are yet not answered. What is the exact relation between Bell's inequalities violation and teleportation? Are there states which violate Bell's inequalities but which cannot be used for teleportation? Is every mixed state that cannot be expressed as a mixture of product states useful for teleportation? One of the main difficulties in answering these questions is that it is not yet known which mixtures violate Bell's inequalities. If for pure states this problem is completely solved—every pure entangled state of any number of arbitrary systems violates Bell's inequalities —for mixtures not even the simplest case (two spin-$\frac{1}{2}$ particles) is solved.

Bell's inequalities and local hidden variable models, as usually defined, refer to joint probabilities of the results of *standard* quantum measurements performed on space separated systems. In case of spin-$\frac{1}{2}$ particles they refer to measurements of spin along different directions (Stern-Gerlach measurements). But there are many other things which can be done to spin-$\frac{1}{2}$ particles. They could be measured in conjunction with other particles, as in teleportation, or one can perform "weak" measurements [7], etc. To be truly separable, two systems separated in space must be described by a local hidden variables model which determines the results of *every* possible experiment. For the case of entangled pure states such sophistication is not necessary—local hidden variable models fail to account even for the joint probabilities of standard quantum measurements. On the other hand, for the mixture considered by Werner, local hidden variables can simulate the joint probabilities of spin measurements. It is then relevant to ask if the model can be enlarged to account for the results of every possible experiment or if there are some generalized Bell type inequalities which are violated. It is therefore still an open question whether teleportation is equivalent or not with violation generalized Bell inequalities.

I hope that the present result will lead to a deeper understanding of this fascinating subject which is quantum nonlocality.

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