Relation between the Correlation Dimensions of Multifractal Wave Functions and Spectral Measures in Integer Quantum Hall Systems

Bodo Huckestein

Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08540

Ludwig Schweitzer

Physikalisch-Technische Bundesanstalt, Bundesallee 100, 38116 Braunschweig, Germany
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We study the time evolution of wave packets of noninteracting electrons in a two-dimensional disordered system in strong magnetic field. For wave packets built from states near the metal-insulator transition in the center of the lowest Landau band we find that the return probability to the origin $p(t)$ decays algebraically, $p(t) \sim t^{-D_2/2}$, with a nonconventional exponent $D_2/2$. D_2 is the generalized dimension describing the scaling of the second moment of the wave function. We show that the corresponding spectral measure is multifractal and that the exponent $D_2/2$ equals the generalized dimension $\widetilde{D}_2$ of the spectral measure.

Wave functions of critical states at a mobility edge separating extended from localized states can be analyzed as multifractals [1–4]. This also holds for quantum Hall systems, where the energy range of the extended states shrinks to a singular point in the center of the Landau bands [5–7]. These critical states exhibit a universal multifractal behavior that can be described by an infinite set of generalized dimensions D_q or equivalently by a singularity strength distribution $f(\alpha)$ [8–10]. As a consequence of the multifractality of the wave functions, the two-particle spectral function $S(k,\omega)$ in the quantum Hall system shows nonconventional behavior [11,12]. At large values of k^2/ω the diffusion constant as a function of wave vector k and frequency ω is reduced from its conventional value D_0, $D(k^2/\omega) \sim D_0 (k^2/\omega)^{-\eta/2}$, with $\eta = 0.38 \pm 0.04$ [12]. The exponent η is related to the generalized dimension D_2 of the wave function by $\eta = 2 - D_2$ [8,13]. The reduction in the diffusion constant will influence the long-time behavior of autocorrelations of wave packets built from states close to the critical energy.

Another class of systems with multifractal wave function are quasiperiodic systems like the self-dual Harper's equation [14,15]. In contrast to the Anderson transition where the density of states is smooth these systems have multifractal spectra. For quantum systems with Cantor spectra it was found that the temporal autocorrelation function $C(t)$ of wave packets built from the multifractal eigenstates exhibits a slow algebraic decay, $C(t) \sim t^{-\delta}$ [16]. Ketzmerick *et al.* showed that δ equals the generalized dimension $\widetilde{D}_2$ of the spectral measure introduced by the local density of states that form the weights in the wave packets.

In this paper we establish a connection between the multifractal properties of the wave functions and the spectral measure for wave packets built from states near the critical energy in the center of the lowest Landau band, namely, that the generalized dimensions D_2 of the

wave function and $\widetilde{D}_2$ of the spectrum are related by $\widetilde{D}_2 = D_2/2$. In order to obtain this result we study numerically the time evolution of the wave packets for finite systems. We find that the temporal autocorrelation function of these wave packets shows a slow algebraic decay $C(t) \sim t^{-\delta}$, with $\delta = 0.81 \pm 0.02$. Conventional, diffusive behavior would result in $\delta = d/2 = 1$, with $d = 2$ the Euclidean dimension of the space. We then show analytically that the multifractal structure of the wave functions leads to the novel form $\delta = 1 - \eta/2 = D_2/2$. By numerically calculating the generalized dimension $\widetilde{D}_2$ of the spectral measure we show that, even in the presence of disorder where the global density of states becomes smooth, the spectral measure introduced by the local density of states near the critical energy is multifractal and that as for Cantor spectra $\delta = \widetilde{D}_2$. This allows us to connect the generalized dimensions of the wave function and the spectral measure with the equality $\widetilde{D}_2 = D_2/2$.

The model that we use for the quantum Hall system is a two-dimensional tight-binding model on a square lattice with on-site disorder and transfer to nearest neighbors only

$$\mathcal{H} = \sum_m \epsilon_m c_m^\dagger c_m + \sum_{m \neq n} V_{mn} c_m^\dagger c_n. \tag{1}$$

The effect of the magnetic field is incorporated by a Peierls substitution [17] via the vector potential $\mathbf{A}(\mathbf{r})$ in the hopping matrix elements

$$V_{mn} = V \exp\left(-(ie/\hbar) \int_{\mathbf{r}_m}^{\mathbf{r}_n} d\mathbf{r}\, \mathbf{A}(\mathbf{r}) \right). \tag{2}$$

We used a system size of 125 by 125 lattice sites with periodic boundary conditions and a commensurate magnetic field of 1/5 flux quanta per unit cell of the lattice. This corresponds to a ratio of system size to magnetic length L/l_c of about 140. The disorder potentials $\{\epsilon_m\}$

were taken from a constant distribution of independent random variables with $-W/2 \le \epsilon_m \le W/2$. The strength of the disorder $(W = 3V)$ was weak enough so that the tight-binding band split into well separated Landau subbands.

In order to observe the multifractal properties of the integer quantum Hall transition the system has to be effectively at the critical point in the center of the Landau band. Approaching the critical energy E_c in the center of the Landau band the localization length $\xi(E)$ diverges. In a finite system the transition is rounded and the system is effectively critical when the localization length $\xi(E)$ of the states under consideration is large compared to the system size L. Over the energy interval used in our calculation the localization length exceeds the system size by at least a factor of 2 [5,6]. Within this critical region no systematic energy dependence of the generalized dimensions was observed.

We used 330 states (about 10% of a Landau band) near the center of the lowest Landau level to build the wave packets. We construct normalized wave packets $\psi(\mathbf{r}, t)$ from the eigenstates $\phi_i(\mathbf{r})$ with energy E_i so that at time $t = 0$ they are centered around $\mathbf{r} = \mathbf{0}$,

$$\psi(\mathbf{r}, t) = A^{-1/2} \sum_i \phi_i^*(\mathbf{0})\phi_i(\mathbf{r})e^{iE_i t/\hbar}, \qquad (3)$$

with $A = \sum_i |\phi_i(\mathbf{0})|^2$. The probability density $p(t)$ to find the particle at site $\mathbf{r} = \mathbf{0}$ at time t is then given by

$$p(t) = |\psi(\mathbf{0}, t)|^2 = A^{-1} \sum_{i,j} |\phi_i(\mathbf{0})|^2 |\phi_j(\mathbf{0})|^2 e^{i(E_i - E_j)t/\hbar}. \qquad (4)$$

This quantity can also be interpreted (up to a constant) as the probability to find the wave packet $\psi(\mathbf{r}, t) = \langle \mathbf{r} | \psi(t) \rangle$ at time t in the initial state $|\psi(0)\rangle$, $p'(t) = A^{-1} p(t) = |\langle \psi(t = 0)|\psi(t)\rangle|^2$. A temporal autocorrelation function $C(t)$ was defined by Ketzmerick et al. by smoothing $p'(t)$ [16]:

$$C(t) = \frac{1}{t} \int_0^t dt' \, p'(t'). \qquad (5)$$

As can be seen from Fig. 1 $C(t)$ decays algebraically, $C(t) \sim t^{-\delta}$, with $\delta = 0.81 \pm 0.02$. Conventional diffusion would lead to a wave packet with asymptotically Gaussian shape and a probability $p'(t) \sim t^{-d/2}$, where d is the Euclidean dimension of the space. The initial deviations from the power law decay seen in Fig. 1 are due to the intrinsic width of the wave packet at $t = 0$, whereas the behavior at times $t > 8000$ arises from finite size effects.

The slow decay of the temporal autocorrelation function is a result of the multifractal structure of the wave functions and is related to the structure of the two-particle spectral function $S(k, \omega)$ [12],
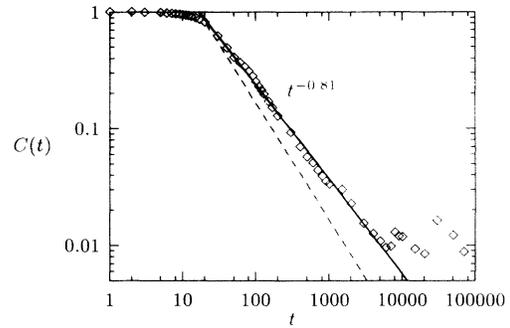
FIG. 1. Temporal autocorrelation function $C(t)$ vs time t (in units of $\hbar/V$) showing the nonconventional power law behavior $C(t) \sim t^{-\delta}$ with $\delta = 0.81 \pm 0.02$. For comparison the dashed line shows the conventional behavior $C(t) \sim t^{-1}$.

$$S(k, \omega) = \rho(E_c)\frac{\hbar}{\pi}\frac{k^2 D(k^2/\omega)}{\omega^2 + \hbar^2 k^4 [D(k^2/\omega)]^2}. \qquad (6)$$

In the limit k, ω going to zero and k^2/ω large compared to the density of states at the critical energy $\rho(E_c)$, $S(k, \omega)$ is proportional to $\omega^{-\eta/2}k^{-2+\eta}$ [12]. The exponent η is related to the generalized dimension D_2 of the wave function by $\eta = 2 - D_2$ [8]. Fourier transforming the spectral function and taking the limit $\mathbf{r} \to \mathbf{0}$ gives the probability density $p(t)$,

$$p(t) = \frac{1}{\rho(E_c)} \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \int_{1/L}^{1/l_m} dk \, k S(k, \omega), \qquad (7)$$

where the momentum integral is cut off at small wavelengths by the microscopic length l_m (i.e., magnetic length or lattice spacing) and at long wavelengths by the system size L. The long time limit of Eq. (7) is governed by the small frequency behavior of the momentum integral which in turn is determined by the large k^2/ω limit of $S(k, \omega)$,

$$\int_{1/L}^{1/l_m} dk \, k S(k, \omega) = \int_{1/\omega L^2}^{1/\omega l_m^2} d(k^2/\omega) \frac{\omega}{2} S(k, \omega)$$

$$\sim \int_{1/\omega L^2}^{1/\omega l_m^2} d(k^2/\omega) \frac{\omega}{2} \omega^{-\eta/2} k^{-2+\eta}$$

$$\sim \omega^{-\eta/2}, \qquad (8)$$

so that in the limit $t \to \infty$ the probability density $p(t)$ becomes

$$p(t) \sim \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} |\omega|^{-\eta/2} \sim \frac{1}{t^{1-\eta/2}} = \frac{1}{t^{D_2/2}}. \qquad (9)$$

Thus the temporal autocorrelations decay as if the wave functions would show conventional diffusive behavior but in a fractal D_2-dimensional space instead of the Euclidean two-dimensional space. This interpretation is
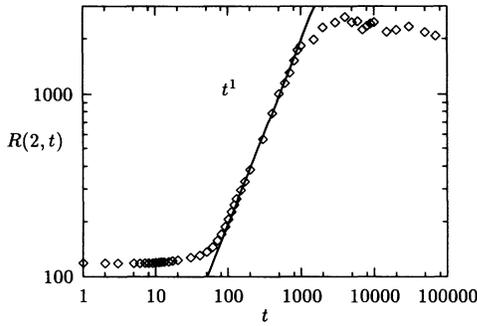
FIG. 2. The variance $R(2,t)$ of the wave packet as a function of time t showing conventional diffusive growth proportional to t^κ with an exponent $\kappa = 1.0 \pm 0.04$.

further supported by observing that the slow decay of the probability density $p(t)$ is solely due to the shape of the wave packet becoming non-Gaussian. The variance of the wave packet still grows proportional to t as is the case for solutions to the diffusion equation in arbitrary dimension. Figure 2 shows the variance $R(2,t) = \int d^2r|\mathbf{r}|^2|\psi(\mathbf{r},t)|^2$ of the wave packet as a function of time t.

The local density of states $|\phi_i(0)|^2$ used as the weights of the eigenfunctions $\phi_i(\mathbf{r})$ in constructing the wave packet introduces a spectral measure. Because of the eigenfunction correlations this measure is multifractal even though the global density of states is noncritical and smooth near the metal-insulator transition.

While the present calculation explains the slow decay of temporal autocorrelations in terms of the multifractal properties of the wave function, Ketzmerick *et al.* have related this phenomenon in quantum systems with Cantor spectra to the multifractal properties of the spectral measure [16]. They showed that δ is given by the generalized dimension $\widetilde{D}_2$ of the spectral measure. We calculated $\widetilde{D}_2$ for the same energy interval that we used for the wave packets from [16]

$$\gamma(\varepsilon) = \sum_i \left(A^{-1} \sum_{E \in \Omega_i(\varepsilon)} |\phi_E(0)|^2 \right)^2 \sim \varepsilon^{\widetilde{D}_2} \quad (\varepsilon \to 0),$$

(10)

where the energy interval is partitioned into boxes $\Omega_i(\varepsilon)$ of width ε.

Figure 3 shows that even in the presence of disorder the spectral measure at the center of the lowest Landau band is multifractal with a generalized dimension $\widetilde{D}_2 = 0.8 \pm 0.05$. Thus the relation $\delta = \widetilde{D}_2$ holds for the quantum Hall system, too. This allows us to directly relate the generalized dimensions D_2 of the wave function and $\widetilde{D}_2$ of the spectral measure, $\widetilde{D}_2 = D_2/2$. This is to our knowledge the first time that such a direct connection could be made.

The correlation dimension D_2 was obtained [18] from
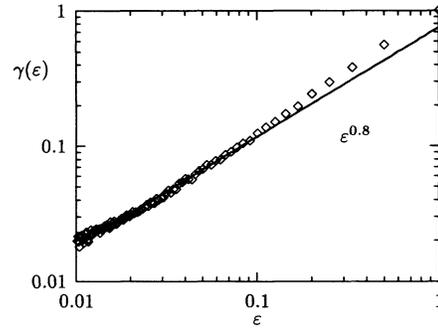
FIG. 3. The scaling of the second moment $\gamma(\varepsilon)$ of the spectral measure in the vicinity of the transition [Eq. (10)] with fractal exponent $\widetilde{D}_2 = 0.8 \pm 0.05$ that coincides with the exponent δ of Fig. 1 within the statistical errors.

the scaling of $P_2(\lambda) \sim \lambda^{D_2}$, see Fig. 4, where $l = \lambda L$ is the length of the boxes $\Omega_i(\lambda)$ used to cover the fractal eigenstate with energy E and

$$P_2(\lambda, E) = \sum_i \left(\sum_{\mathbf{r} \in \Omega_i(\lambda)} |\phi_E(\mathbf{r})|^2 \right)^2.$$

(11)

In conclusion, we have established a relation between the multifractal properties of wave packets built from critical eigenstates near the center of the lowest Landau level of a quantum Hall system and the multifractal properties of the corresponding spectral measure. Specifically we find that the generalized dimensions D_2 of the wave function and $\widetilde{D}_2$ of the spectral measure are related by $\widetilde{D}_2 = D_2/2$. As a consequence of these multifractal properties, the temporal autocorrelation function $p(t)$ decays with a nonconventional exponent $\delta = D_2/2$. Thus the probability for the wave packet to stay near the origin after long times is greatly enhanced compared to conventional diffusion in two dimensions. This behavior can be interpreted as a wave packet showing conventional dif-
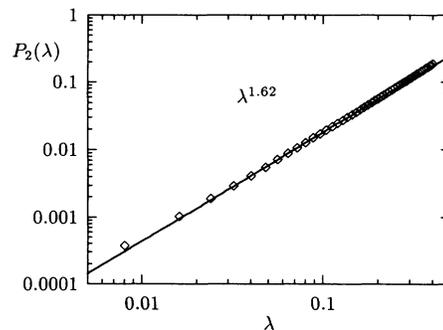
FIG. 4. The scaling of the second moment $P_2(\lambda) \sim \lambda^{D_2}$ of critical states near the transition with the correlation dimension $D_2 = 1.62 \pm 0.02$ fulfilling the relation $\widetilde{D}_2 = D_2/2$ within the statistical errors.

fusion on a D_2-dimensional fractal.

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