Exact Solution for Superfluid Film Vortices on a Torus

Andres Corrada-Emmanuel
Department of Physics and Astronomy, Swarthmore College, Swarthmore, Pennsylvania 19081
(Received 27 September 1993)

The problem of describing the velocity field of quantized vortices on a torus is solved exactly by use of Riemann's bilinear relations of 1857. The solution is used to discuss the behavior of superfluids on porous media. The close connection between topology and quantization of circulation is emphasized as well as its implications for a successful theory of the experiments.

PACS numbers: 67.40.Vs, 67.40.Bz, 67.40.Hf

Since the success of the Kosterlitz-Thouless (KT) theory of superfluid destruction on the plane in the late 70s there has been a renewed effort to understand the superfluid transition in ^{4}He films absorbed on porous materials (packed powders, Vycor, xerogel, aerogel). No theoretical concensus has been reached on how to interpret the experimental results. For example, the dimensionality of the system at the superfluid transition is still debated. Williams and co-workers argue for a two-dimensional transition with KT type behavior for all substrates [1]. Reppy and co-workers along with Chan and co-workers argue that on Vycor the transition is similar to the one in bulk helium and on xerogel and aerogel it is some unknown transition [2]. Machta and Guyer (MG) recognized early on that the connectivity of these porous substrates creates a topologically complex surface (i.e., one with nonzero genus) that modifies the interaction between vortices and creates new ones, the so-called "pore" flows [3]. This modification of the pair interaction was implemented in the models that sought to explain the transition as a modified KT transition by arguing that the interaction energy between vortex pairs was linear with the pair separation at distances longer than the pore diameter [4].

The purpose of this paper is twofold: to present the first exact solution for vortices on a nonzero genus surface, thereby explicitly demonstrating the importance of topology in modifying the film excitations, and to show that for the surface we are considering, the torus, the interaction between pairs goes linearly with their separation only in the case of a single vortex pair. In other words, the exact solution is nonlinear, which invalidates the simple approximation used by MG and Williams.

Consider superfluid film vortices on a plane. Since the superfluid must obey the Feynman-Onsager quantization hypothesis,

$$\oint \mathbf{v}\cdot d\mathbf{l} = n\left(\frac{h}{m}\right), \tag{1}$$

we know that the velocity near a vortex core must be tangential and with a magnitude of

$$|\mathbf{v}| = n\left(\frac{h}{m}\right)\frac{1}{r}, \tag{2}$$

where r is the distance to the core. This flow field can be represented by the complex potential function

$$\phi = n\left(\frac{h}{m}\right)\ln(z), \tag{3}$$

with $z=x+iy$. If we have a collection of vortices then the flow potential can be immediately written down as a sum of the individual potentials. A sum of these individually quantized flows is also quantized. The proof is simple: By using the Cauchy residue theorem we can see that any contour integral will have a magnitude that depends on the sum of the residues enclosed. Since we set these residues to individually satisfy the quantization condition, any sum of them will also obey it. This argument is familiar to cognoscenti of the Kosterlitz-Thouless transition. But it does not work in a multiply connected surface like those of the porous materials. It assumes a topological condition that is not satisfied in general: A simple closed curve divides the plane into two regions, the inside and the outside (Jordan's curve theorem). In general a vortex on a multiply connected surface that is locally described by the same potential as the plane vortex, and therefore satisfies the quantization condition around any path that encloses the core, is not guaranteed to lead to quantized flows around other paths. An example on the torus will make this clear.

Consider two vortices of unit quantization on opposite ends of a torus (see Fig. 1). As stated above, by picking

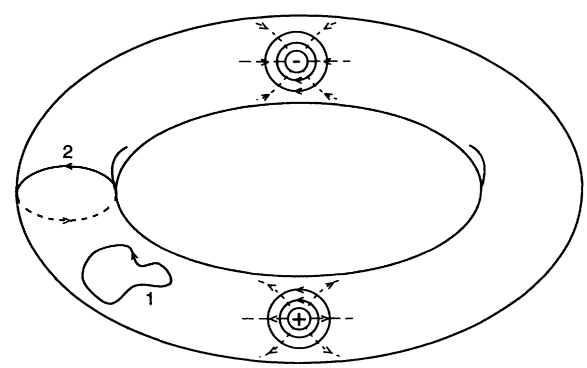

FIG. 1. Torus with two film vortices on its surface.

0031-9007/94/72(5)/681(4)$06.00

the strengths of the vortex potentials to be an integer of the quantum of circulation, we can guarantee that the circulation on paths of type 1 is quantized. But paths of type 2 do not enclose either vortex. What is the circulation about them? We can quickly calculate it by exploiting the similarity of planar vortices to two-dimensional Coulomb charges. This equivalence means that a calculation of circulation for vortices is the same as a flux calculation for charges. Therefore, by noting that in this symmetric arrangement each charge will have half its flux on either side of the torus, we conclude that the vortices create half a quantum of circulation on paths of type 2. This failure is remedied by adding pore flows around the torus with half a quantum of circulation. This example exhibits what is required for writing down a quantized flow on the torus: Vortices that locally look like quantized planar vortices must have pore flows added to guarantee quantized circulation on all paths. Clearly, none of these arguments are necessary when discussing a classical fluid flow. Thus, the problem of constructing quantized vortex flows on a multiply connected surface serves as another illustration of the close relationship between topology and quantum mechanics.

To construct the general solution we first do a conformal mapping of the torus into a rectangle. The surface of the torus (Fig. 2) is described by the equations

$$x=\rho\cos\phi,\quad y=\rho\sin\phi,\quad z=r\sin\alpha\,, \tag{4}$$

with ρ given by the two radii of the torus as

$$\rho=R+r\cos\alpha\,. \tag{5}$$

By defining new coordinates

$$\xi=\phi,\eta=\int_0^\alpha \frac{r\,d\theta}{R+r\cos\theta}\,, \tag{6}$$

we map the torus into a so-called "normal" rectangle with a width of 2π and height of $2p$, where p is given as

$$p=\pi\frac{r/R}{\sqrt{1-(r/R)^2}}\,. \tag{7}$$

In this coordinate system we can write the vortices that are isomorphic to Coulomb charges by employing the complex variable

$$z=\xi+i\eta \tag{8}$$

and using elliptic functions. The general solution to a locally $1/z$ field on the normal rectangle is of the form

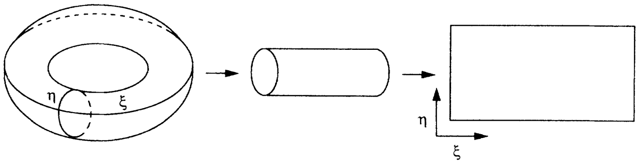

FIG. 2. Turning the torus surface into a "normal" rectangle.

$$A+\sum_k q_k\zeta(z-z_k)\,, \tag{9}$$

with A a complex constant as yet undetermined, and we have written the Weierstrass zeta function $\zeta(z;2\pi,i2p)$, defined on the normal rectangle, in an abbreviated fashion [5]. Requiring that this field represent a Coulomb field (i.e., no curl for the vector field) fixes the constant A and we get an expression for the Coulomb field,

$$E_\xi-iE_\eta=\sum_k q_k\left(\frac{2\eta_1}{2\pi}\xi_k+\frac{2\eta_2}{2p}\eta_k+\zeta(z-z_k)\right). \tag{10}$$

The η constants, η_1 and η_2, are given by evaluating the Weierstrass zeta function at the half periods, i.e., $\eta_1=\zeta(\pi)$ and $\eta_2=\zeta(ip)$. To obtain the flow for the fluid we would multiply this by i and change q_k to $n(h/m)$ to obtain the velocity field $v_\xi-iv_\eta$.

But we know that this solution may not have quantized circulation. How much pore flow do we have to add? The symmetric placement of the two vortices in our earlier example made it easy to guess the answer. In general we would have to calculate the appropriate line integrals along the ξ and η directions. Here is where we can use a remarkable theorem proved by Riemann in 1857 [6]. The original intent of the theorem was to relate Abelian differentials of the first and third kind. These relations are called bilinear relations and while they apply in general to any compact surface of arbitrary genus, in the case of the torus they allow us to calculate the circulation for any vortex arrangement in terms of the vortex positions. For our purposes the theorem can be stated as follows:

$$\int_\xi \mathbf{v}\cdot d\mathbf{l}=-\left(\frac{h}{m}\right)\sum_k n_k(\eta_k/2p)\,, \tag{11}$$

$$\int_\eta \mathbf{v}\cdot d\mathbf{l}=+\left(\frac{h}{m}\right)\sum_k n_k(\xi_k/2\pi)\,. \tag{12}$$

This is a physically satisfying solution since the sum terms can be interpreted as the "topological dipole" of the vortices. We coin this new term to underline the fact that the result of the summation is a dimensionless quantity that is independent of the size of the original torus and depends solely on the dimensionless variable p. It is also clear that it arises because of the nontrivial topology of the torus.

The construction of quantized vortex solutions is now possible. Since we can always add quantized pore flows to an already quantized flow the answer cannot be unique, so let us construct the minimum energy flow. The Coulomb vortices we have solved for and the pore flows are orthogonal vector fields over the torus [7]. Therefore, the minimum pore flow needed in any arrangement of vortices will always have half or less a quantum of circulation. For example, the earlier arrangement of vortices symmetrically placed around the torus corresponds to the case $\xi_1=0$, $\eta_1=0$, $\xi_2=\pi$, $\eta_2=0$.

So Riemann's bilinear relations tell us

$$\int_\eta \mathbf{v}\cdot d\mathbf{l} = \tfrac{1}{2}(h/m)\,, \tag{13}$$

which is what we deduced before from symmetry considerations. The minimum energy flow would either add or subtract a pore flow in the η direction with half a quantum of circulation.

The implications of the exact solution for more realistic models of porous substrates are as follows. The first is the reaffirmation of the observation by MG that the topology of the substrate couples Coulomb vortices to the pore flows because of the constraint of quantized circulation. However, implementing this insight in a realistic model of the substrates, like the "jungle-gym" lattice (see Fig. 3), is not easy. As stated before, the approach used so far is to approximate the vortex interactions by a linear term, the so-called "string picture." The solution presented here for the torus exhibits this effect for a single Coulomb vortex pair because the topological dipole of the pair is proportional to their separation. This requires the addition of a pore flow with energy proportional to the pair separation if the flow is to be quantized. The case of multiple pairs, however, cannot be treated this way because the topological dipole will not be linear in the pair separations. In other words, the presence of other vortex pairs affects the circulation about the nonenclosing paths and therefore the pore flows needed to create the quantized flow of lowest energy are not the simple sum of what is required for the individual pairs. This is in marked contrast to the case of the plane where the quantization of Coulomb vortices is guaranteed by setting the correct "charge." The implication for a KT type model is obvious since such a scheme involves integrating out the effect of vortex pairs separated by less than the length scale of the current iteration. Renormalizing out vortex pairs that always have an energy linear in their separation is selecting out a subset of all possible vortex arrangements that satisfy quantization of circulation. This feature must be justified before this approximation can be considered to capture the essential physics of the superfluid transition in porous media.

The second lesson concerns the dispute over the correct model to use for the films at the superfluid transition. Clearly nothing can be said in the context of this paper about the specific details of transition (critical exponents, etc.), but the following observation can be made about comparing results in packed powders with the other substrates. There are two dimensionless measures of length for vortices on the torus. One is set by the torus itself via the dimensionless variable p, the other is given by a_0/r where a_0 is the size of a vortex core. Restricting ourselves to small values for this latter measure, we can see that the energy of a vortex pair is determined by p. This parameter can vary from zero to infinity. A value of p close to zero represents a skinny torus, like a bicycle tire. A value of p close to infinity represents a fat torus, like a bagel. Packed powders are made up of grains that we can take to be essentially spherical. They can be expected to create pores that are akin to the fat torus limit because of the narrow contact points between kissing spheres. It is interesting to note that the large p limit yields the same solution as the cylinder case solved by MG. In our case, however, the cylinder exists in the dimensionless space of the variables ξ and η. In contrast, a substrate like Vycor does not have these "kissing" points. Using a value of 0.30 for the porosity of Vycor and assuming that Vycor can be described by the jungle-gym model of MG, we arrive at a value of p of order unity [8]. This is on the side of the thin torus limit. Therefore, packed powders and Vycor are located in different regions of the p-parameter space. Stated differently, the ratio of the distance between intersections of the pores and their diameter is different in the two materials. This is crucial because of the constraint of quantized circulation. The effect of a large p is to make it difficult for pore flows with circulation around the small contact points to be created since their energy is proportional to p itself. We speculate that this feature may explain the difference between the experimental results in packed powders versus materials like Vycor.

In conclusion, we have shown that the interaction between film vortices in topologically complex substrates is nontrivial because of the quantization of circulation. An explicit solution in the case of the torus was presented. The approximation that vortex pairs have an energy proportional to their separation was shown to be incorrect for multiple vortex pairs. We speculated that the dependence of this solution on the parameter p may affect the nature of the superfluid transition in different substrates.

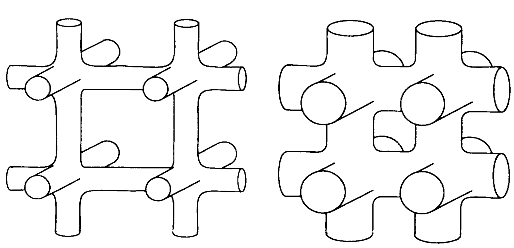

FIG. 3. "Jungle-gym" model of porous materials: (a) Low-p lattice; (b) high-p lattice.

[1] G. Williams, in *Excitations in 2D and 3D Quantum Fluids*, edited by A. F. G. Wyatt and H. J. Lauter (Plenum, New York, 1991).
[2] J. D. Reppy, J. Low Temp. Phys. **87**, 205 (1992).
[3] J. Machta and R. A. Guyer, Phys. Rev. Lett. **60**, 2054 (1988).
[4] F. Gallet and G. Williams, Phys. Rev. B **39**, 4673 (1989).

[5] E. J. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge Univ. Press, Cambridge, 1958), 4th ed.
[6] G. Springer, *Introduction to Riemann Surfaces* (Chelsea, New York, 1981), 2nd ed.
[7] This is known as Hodge's theorem and states that the incompressible flows of a film on a Riemann surface can be decomposed into three orthogonal parts: the pore flows, the flow associated with sources or sinks, and the vortices conjugate to Coulomb charges. Note that this theorem was incorrectly stated in MG [Ref. (3), Eq. (2)] where the source or sink space was erroneously identified as describing "spin waves." See Ref. [6] for more details on the theorem.
[8] P. Levitz, G. Ehret, S. K. Sinha, and J. M. Drake, J. Chem. Phys. **95**, 6151 (1991).