Controlling Spatiotemporal Chaos in Coupled Map Lattice Systems

Hu Gang[1,2,3] and Qu Zhilin[2]
[1] *Center of Theoretical Physics, Chinese Center of Advanced Science and Technology (World Laboratory), Beijing 8730, China*
[2] *Department of Physics, Beijing Normal University, Beijing 100875, China**
[3] *Institute of Theoretical Physics, Academia Sinica, Beijing, China*
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A spatiotemporal system is modeled by a coupled map lattice. Feedback pinnings are used to control chaos of the system by stabilizing a certain unstable reference state. As the pinning distribution is dense enough, the unstable state can be stabilized. If the density is low, the reference state may not be approached asymptotically. In this case, however, the pinnings can still effectively suppress chaos and produce rich spatiotemporal structures. If a solution is locally stable while the transient process to this state is extremely long and chaotic, pinnings of very low density can well control the transient chaos.

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Recently, controlling chaos has become an active field in the study of nonlinear dynamics [1–9], due to the great potential of applications. However, up to date the research has mainly focused on the low-dimensional systems. Actually, controlling chaos (or controlling turbulence) in spatiotemporal systems is even more important. For instance, controlling chaos or turbulence in plasma devices, laser systems, chemical reactions, etc., is of practical importance, where both spatial and temporal dependences should be considered. To control chaos in these systems, one has to deal with the problem of controlling the total systems by modulating very few freedoms. A very simple idea in this regard is to put some local controlling (pinnings) in space. Therefore, it is of crucial importance to study how dense the pinnings should be for controlling chaos, and how the pinnings influence the dynamics of the system. These two problems will be the main concerns of the present Letter.

Roughly speaking, there are two ways for controlling chaos: a feedback control [1–6] and a nonfeedback one [7–9]. In the former case, one takes a given unstable reference orbit, which is a trajectory of the unperturbed system, as the goal of the controlling. The controlling input is very small when the system is well controlled. On the contrary, the latter is not related to a certain particular trajectory. The aim of controlling is to suppress chaos. Therefore, the controlling input does not vanish as the system is under control, and the dynamical structure of the controlled system may be considerably different from the original system. In this Letter we consider a well known coupled map lattice (CML) model [10–14]. We start from the feedback control to see how dense the pinnings should be for controlling the system to certain reference state. As the pinning density reduces below a critical value the system can no longer be locked to the given state. However, the pinnings still play an important role to create various interesting spatiotemporal structures, and to effectively suppress chaos.

Here we consider a one-dimensional CML model [11–14]:

$$x_{n+1}(i) = (1-\epsilon)f[x_n(i)] + \tfrac{1}{2}\epsilon\{f[x_n(i-1)] + f[x_n(i+1)]\}, \qquad (1)$$

where $i = 1, 2, ..., L$ are the lattice sites, and L the system size. In this Letter, the periodic boundary condition, $x_n(i+L) = x_n(i)$, is assumed. Moreover, we take $f(x) = ax(1-x)$. With single site ($L = 1$), model (1) reduces to the well known logistic map, which has a period-doubling cascade with the accumulation point at $a_c = 3.5699456...$, and chaos can be found in the regime $a_c < a \leq 4$. The dynamical behavior of model (1) with $L > 1$ has been also extensively investigated and quite well understood [10–14]. The coupling terms in (1) introduce very rich spatiotemporal patterns, such as homogeneous stationary solution; inhomogeneous stationary solutions (spatial patterns); homogeneous periodic solutions; and inhomogeneous periodic solutions (running or standing waves). In a stable chaotic state, all these patterns embedded in the chaotic attractor are unstable. In Fig. 1(a) we plot the space-time diagram of the system with $\epsilon = 0.8, a = 4$. The lattice length is taken to be $L = 60$, and the initial condition is prepared as pseudo-random numbers uniformly distributed in the interval $[0, 1]$. Afterwards, we will always take $L = 60$ and use such an initial condition unless specified otherwise. The motion of the system is fully turbulent at these parameter values [11], as clearly shown in Fig. 1(a).

To control this system, we use pinnings defined in the following way:

$$x_{n+1}(i) = (1-\epsilon)f[x_n(i)] + \tfrac{1}{2}\epsilon\{f[x_n(i-1)] + f[x_n(i+1)]\} + \sum_{k=0}^{L/I}\delta(i - Ik - 1)g_n \qquad (2)$$

with

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$$g_n = (1-\epsilon)p_n(i)x_n(i)[x_n(i) - \bar{x}_n(i)] + \tfrac{1}{2}\epsilon\{p_n(i-1)x_n(i-1)[x_n(i-1) - \bar{x}_n(i-1)] + p_n(i+1)x_n(i+1)[x_n(i+1) - \bar{x}_n(i+1)]\}\,, \tag{3}$$

where I is the distance between two neighboring pinnings, $\bar{x}_n(i)$ the reference state to be stabilized, $p_n(i)$ the feedback strength added to the ith site, and $\delta(j) = 1$ for $j = 0$ and $\delta(j) = 0$ otherwise. The advantage of this type of feedback is that, on one hand, the nonlinear part effectively avoids the overflow in numerical simulations, on the other hand, in the vicinity of the reference state the controlling is the conventional linear negative feedback. The main idea is to control the entire system by controlling a part of the sites, i.e., by setting pinnings which have spatial density $1/I$.

First we want to control the system by stabilizing the homogeneous stationary state

$$\bar{x}_n(i) = x^* = 1 - 1/a, \tag{4}$$

and an inhomogeneous stationary state

$$\bar{x}_n(2j-1) = \frac{B + \sqrt{B^2 - 4C}}{2a(2\epsilon - 1)}\,,$$
$$\bar{x}_n(2j) = \frac{B - \sqrt{B^2 - 4C}}{2a(2\epsilon - 1)}\,, \quad j = 1, 2, ..., L/2\,, \tag{5}$$

where $B = 1 - a + 2a\epsilon$ and $C = \epsilon - a\epsilon - 2a\epsilon^2$. Both states are unstable for $\epsilon = 0.8$ and $a = 4$ without controlling. For simplicity, we take uniform control, i.e., $p_n(i) = p$. We control state (4) with $p = 3, I = 2$ [Fig. 1(b)], and state (5) with $p = 2, I = 4$ [Fig. 1(c)], respectively. It is worthwhile pointing out that, in both cases, the attracting basins of the reference states (after controlling) are very large. Although the initial conditions are random numbers ranging from 0 to 1, the system finds the desired patterns [i.e., Eqs. (4) and (5) in Figs. 1(b) and 1(c), respectively] very quickly. This fact is very important for practically controlling the system to desirable patterns. As the system reaches the given aim state the feedback vanishes, and then the dynamics of the unperturbed system is not changed.

Obviously, the controlling efficiency depends on the

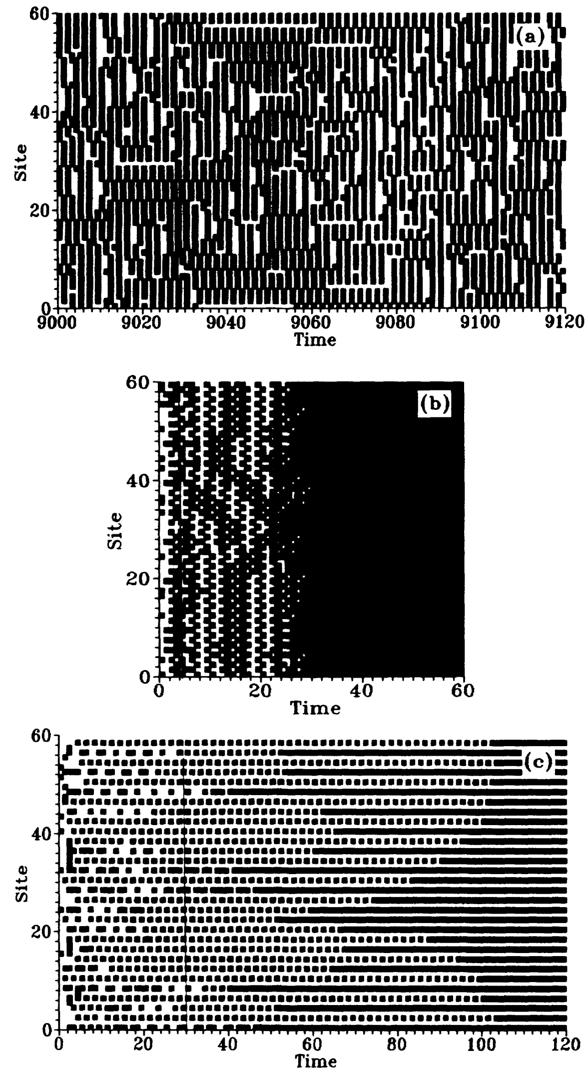

FIG. 1. (a) Space-time diagram for CML (1) with $\epsilon = 0.8, a = 4$, and $L = 60$. Pixels are painted black if $x_n(i) \geq 0.75$, and left blank otherwise. The system dynamics is, obviously, chaotic. (b) Homogeneous unstable stationary state is approached by controlling sites $i = 1, 3, 5, ..., 59$ with $p = 3, \epsilon = 0.8$, and $a = 4$. The figure is plotted in the same manner as (a). (c) The unstable space-period-two pattern, $\bar{x}_n(2j-1) = 0.536\,537...$, $\bar{x}_n(2j) = 0.880\,129...$, $j = 1, 2, ..., 30$, controlled by pinning sites $i = 1, 5, ..., 57$ with $p = 2, \epsilon = 0.8, a = 4$ Pixels are painted black if $x_n(i) \geq 0.88$, and left blank otherwise.

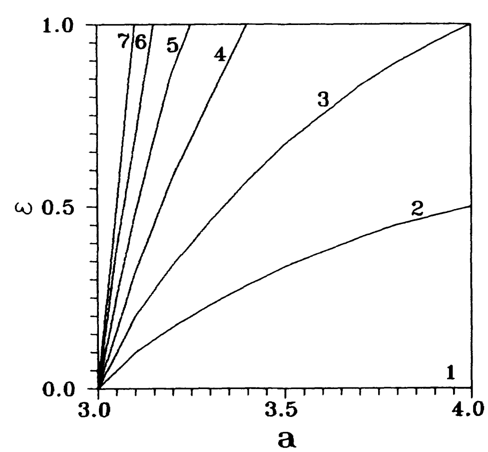

FIG. 2. The controllable regions of the homogeneous stationary state (4) in the a-ϵ plane. The line numbers indicate the reciprocals of the minimal pinning density. Above each line the reference state can be stabilized by the pinnings with the indicated density.

coupling strength ϵ and the nonlinear coefficient a. In Fig. 2 we plot the numerical result of I_m in the a-ϵ plane, where $1/I_m$ is defined as the minimal density of pinnings for stabilizing the homogeneous stationary state. From the figure it is clear that the minimal pinning density can be considerably reduced by decreasing a and increasing ϵ. It seems to us that when $a \to 3$ (period doubling point) the quantity $1/I_m$ may approach zero (for $L \to \infty$, of course) for nonzero ϵ. The physical interpretation of Fig. 2 is clear. The deeper the system sinks into the chaos regime, the more pinnings are needed to suppress chaos and the stronger coupling is needed to force the uncontrolled sites following the discipline of the reference state. We have investigated I_m in the a-ϵ plane for other reference states and similar results are obtained, which will be discussed in another paper.

The unstable structures can be stabilized only for a certain interval of p. As I increases the interval of p for an effective controlling decreases. In the case of $I > I_m$, the pinning density is so low that the reference state cannot be stabilized for any p; then the system can no longer be driven to the reference state by the feedback. Nevertheless, the response of the system to the controlling in this case is still very interesting. The chaotic motion of the unperturbed system can be well controlled, and rather rich spatiotemporal patterns, which are stable against perturbations, can be induced by pinnings. In Fig. 3 we plot $x_n(i)$ versus i for n from 9800 to 10 000, where we again use (4) as the reference state, and take $\epsilon = 0.8$, $a = 4$, and $I = 20$ (a very low pinning density, much lower than I_m!). As pinning strength p increases, chaos of the system is more and more effectively suppressed. One finds successively fully developed turbulence [Fig. 3(a), $p = 0.5$], frozen mixture of standing wave regions and chaotic regions [Fig. 3(b), $p = 1.2$], and perfect standing waves [Fig. 3(c), $p = 2.0$], for the same random initial condition. The chaos level is reduced by increasing p. By varying the initial state, the asymptotic state can be modified and very rich spatiotemporal patterns are observed. However, the main feature of suppressing chaos by increasing p from zero is not changed. For even larger p we are faced again with turbulent motions and overflow of numerical simulations. In Fig. 3 our control can be regarded as nonfeedback control since the reference state is not approached as $t \to \infty$ and the perturbation is finite in the asymptotic state. It is easy to verify that, under the periodic boundary condition and for $L = 20$, the motion of system (1) is fully turbulent at the parameter values $\epsilon = 0.8$ and $a = 4$. Thus, the efficiency of controlling chaos by such rare pinnings is strikingly high.

In recent years the study of transient chaos [11,12,14,15] has attracted much attention due to the practical importance. In a certain case, the system approaches a stable regular motion as $t \to \infty$. However, the transient process is extremely long and chaotic; one can observe only chaotic motion for finite time. The study of transient chaos becomes even more useful in spatiotemporal systems. In CML (1), infinite stable nonturbulent patterns exist in the "period-window" regime; the corresponding attracting basins, however, are usually very small, and the times needed for the system to enter these basins increase exponentially with the system size [11,12,14]. Thus, controlling the transient chaos, enlarging the attracting basin of a given stable state, and shortening the relaxation time towards this aim state are very important tasks in CML systems and other space-time-dependent systems.

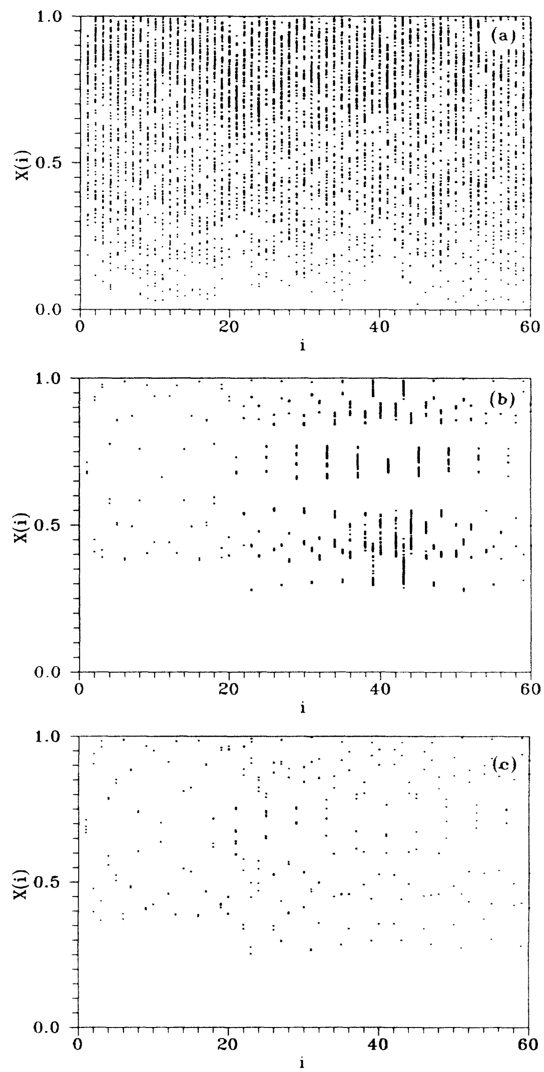

FIG. 3. $x_n(i)$ versus i for n from 9800 to 10 000 with $I = 20$. Pinnings are input at $i = 1, 21, 41$, $L = 60$, $\epsilon = 0.8$, $a = 4$. (a) $p = 0.5$; (b) $p = 1.2$; (c) $p = 2.0$.

At the parameter values $\epsilon = 0.3$ and $a = 4$, system (1) has a running wave solution of time-period-two and space-period-four, which is shown in Fig. 4(a). (Here we take an initial condition very near the running wave state, the system approaches the aim state very quickly since the state is stable.) If we prepare the initial condition as random numbers uniformly distributed in [0, 1], the tran-

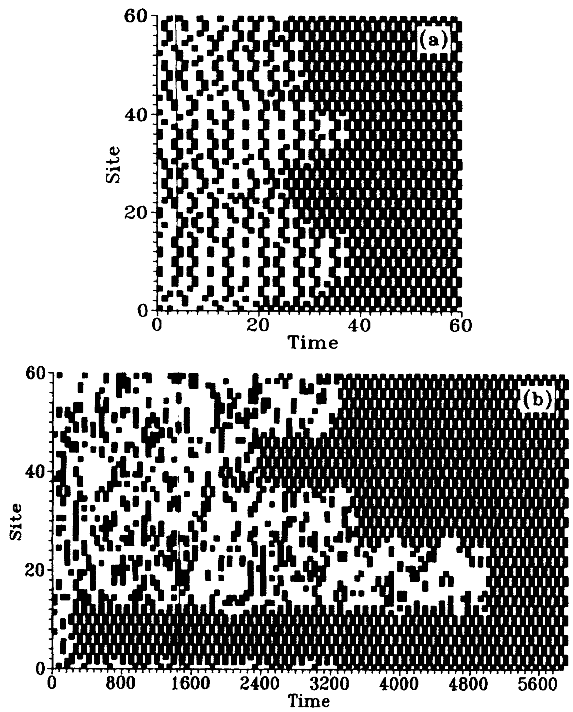

FIG. 4. (a) Time-period-two and space-period-four running wave for $\epsilon = 0.3$ and $a = 4$. The initial condition is prepared as $x_0(i) = \bar{x}_0(i) + \sigma$ with $\bar{x}_0(i)$ being the reference state and σ being random numbers, $|\sigma| \leq 0.04$. Pixels are painted black if $x_n(i) \geq 0.89$, and left blank otherwise. (b) The transient process of the system evolution under controlling. $p_n(i) = 2$ for the sites with $\bar{x}_n(i) = 0.898\,729\,07...$, and $p_n(i) = 1$ for those with $\bar{x}_n(i) = 0.458\,413\,79....$ Pinnings are input at sites $i = 1, 13, 25, 37$, and 49. The initial state is prepared as random numbers uniformly distributed in $[0, 1]$. The figure is plotted in the same manner as (a) except we take data once every 49 iterations.

sient time to the stable running wave state increases very quickly as L increases. For $L = 60$ the average transient time is of order 10^{13} iterations or more [14]. It is practically impossible to reach this state in our computer. The motion in the transient process is purely turbulent, like what is shown in Fig. 1(a). Now we feedback one site for each of the twelve sites in the manner of Eq. (2) with $I = 12$. The pinning density is very low. For a wide range of $P_n(i)$ the transient chaos can be effectively controlled. In Fig. 4(b) we show the transient process of the system under controlling. The relaxation time is about 5000 which is uncomparably shorter than that without controlling. The controlling result is rather robust. We have taken various initial random conditions in the variable space $[0, 1]$, the system always finds almost the same attractor, and realizes the running wave with few defects (in most cases, one or two pairs of defects) but in Fig. 4(b) no defect exists.

It is emphasized that the approach of controlling chaos suggested in this Letter for the CML systems can be used for controlling chaos in partial differential equations. The idea to control the total system by pinning certain local points is expected to be particularly effective for controlling chaos in continuous extended systems; that, however, will be investigated in a separate paper.

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* Mailing address.

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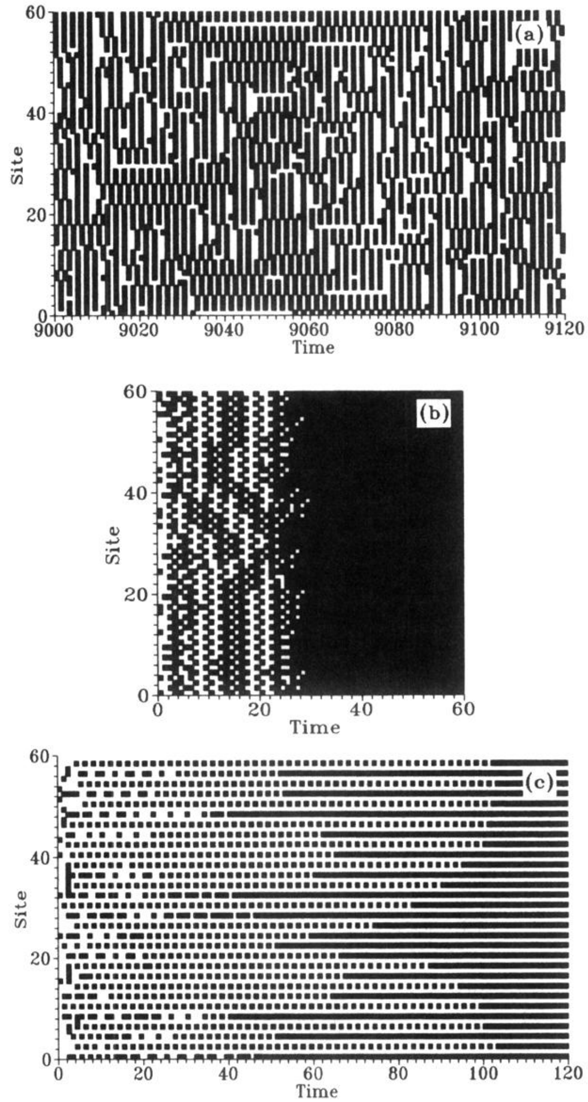

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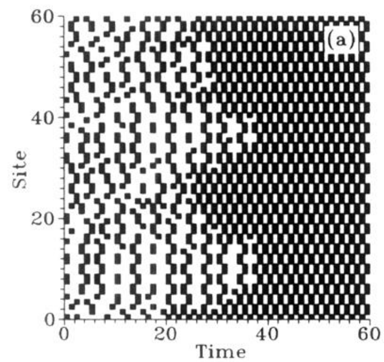

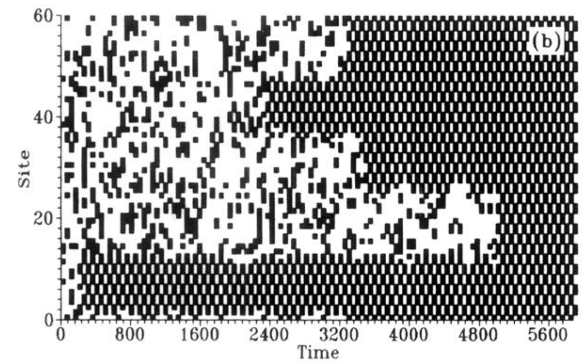

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