Equation of State of an Anyon Gas in a Strong Magnetic Field

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The statistical mechanics of an anyon gas in a magnetic field is addressed. A harmonic regulator is used to define a proper thermodynamic limit. When the magnetic field is sufficiently strong, only exact N-anyon ground states, where anyons occupy the lowest Landau level, contribute to the equation of state. Particular attention is paid to the interval of definition of the statistical parameter $\alpha \in [-1, 0]$ where a gap exists. Interestingly enough, one finds that at the critical filling $\nu = -1/\alpha$ where the pressure diverges, the external magnetic field is entirely screened by the flux tubes carried by the anyons.

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It is now widely accepted that anyons [1] should play a role in the quantum Hall effect [2]. In the case of the fractional quantum Hall effect, Laughlin wave functions for the ground state of N electrons in a strong magnetic field with filling $\nu = 1/m$ provide an interesting compromise between Fermi degeneracy and Coulomb correlations. A physical interpretation is that at the fractional filling electrons carry exactly m quanta of flux ϕ_0 (m odd), $m-1$ quanta screening the external applied field. One is left with usual fermions (i.e., anyons carrying one quantum of flux) in an effective magnetic field with filling 1, or with bosons in a magnetic field entirely screened. Anyons with intermediate statistics $1/m$ enter the game when localized excitations above the ground state are recognized as carrying fractional charge and statistics. The presence of a gap in the spectrum is crucial for explaining the absence of dissipation on the Hall plateaus. In the case of the integer quantum Hall effect, on the other hand, one considers a gas of noninteracting electrons filling exactly n Landau levels. The ground state is not degenerate, one has automatically a cyclotron gap, and the Coulomb interaction can be neglected.

In this Letter we calculate the equation of state of an anyon gas in a strong magnetic field at low temperature. We argue that considering boson based anyons the N-anyon ground-state problem is entirely solvable in terms of known linear states [3,4], which end up being a product of the one-body Landau ground state. Particular care is given to the interval of definition of the statistical parameter $\alpha \in [-1, 0]$ in order that the gap above the ground state is under control. We find that the pressure diverges when the filling factor ν takes its maximal value $\nu = -1/\alpha$, suggesting that everything happens as if at most $-1/\alpha$ anyons can occupy a given one-body Landau ground state. (After completion of this work, we noticed that a similar conclusion has been reached in [5], by a qualitative scaling argument using one-component plasma analogy.) At the critical value of the filling factor, the anyon gas completely screens the external applied magnetic field, leaving a free Bose gas. Moreover, the system is incompressible (both nondegenerate and with a gap).

Let us consider in the symmetric gauge the Hamiltonian of N anyons (charge e, flux ϕ) in a constant magnetic field B ($\mathbf{k}$ is the unit vector perpendicular to the plane, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$),

$$H_N = \sum_{i=1}^{N} \frac{1}{2m}\left(\mathbf{p}_i - \alpha \sum_{j\neq i} \frac{\mathbf{k}\times\mathbf{r}_{ij}}{r_{ij}^2} - e\frac{B}{2}\mathbf{k}\times\mathbf{r}_i\right)^2 . \quad (1)$$

The statistical parameter, $\alpha = e\phi/2\pi$, measures the algebraic fraction of quantum of flux $\phi_0 = 2\pi/|e|$ carried by each anyon. One deals with boson based anyons, meaning that the wave functions ψ are symmetric. The anyons are coupled to the external magnetic field by their electric charge e. Coulomb interactions between anyons are ignored. This will be justified *a posteriori* when the anyon gas will be taken at its critical filling where the ground state is nondegenerate and has a gap. The Hamiltonian being invariant under $(x_i, y_i, \alpha, \epsilon) \to (x_i, -y_i, -\alpha, -\epsilon)$, where $\epsilon = eB/|eB|$, the spectrum and thus the partition function are invariant under $(\alpha, \epsilon) \to (-\alpha, -\epsilon)$. They depend only on $|\alpha|$, $\epsilon\alpha$, and $\omega_c = |eB|/2m$ (half the cyclotron frequency). One chooses $\epsilon = +1$; in the opposite case one would simply change $\alpha \to -\alpha$. The shift $\alpha \to \alpha + 2$ is equivalent to the regular gauge transformation $\psi \to \exp(-2i\sum_{i<j}\theta_{ij})\psi$, which does not affect the symmetry of the wave functions. The spectrum is thus periodic in α with period 2.

What is the N-anyon ground state in a magnetic field? Let us reexamine this question more closely by paying particular attention to the domain of definition of α. It will turn out that α has to be taken in the interval $[-1, 0]$, implying that the magnetic field is antiparallel to the flux tubes carried by each anyon. (Because of the periodicity $\alpha \to \alpha + 2$, opposite direction has to be understood to a given even number quanta of flux.)

One has the ground-state basis [3,4] (in complex coordinate $z_i = x_i + iy_i$)

$$\psi = z^l \prod_{i<j} r_{ij}^{-\alpha} \prod_{i<j} z_{ij}^{m_{ij}} \exp\left(-\frac{m\omega_c}{2}\sum_i^N z_i\bar{z}_i\right),$$

$$l \geq 0, \quad m_{ij} \geq \alpha, \quad (2)$$

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where $z = \sum_i z_i/N$ is the center of mass coordinate. The total angular momentum is $L = l + \sum_{i<j} m_{ij}$. Clearly, the m_{1j}'s can be chosen as independent orbital quantum numbers. Moreover, since a bosonic representation has been chosen, the eigenstates (2) must be symmetrized, leading to additional constraints on the m_{ij}'s.

If α is not an integer, the eigenstates (2) are entirely contained in the class I [4] of eigenstates of the N-anyon problem. If α is an integer α_0, the $m_{ij} > \alpha_0$ states are contained in class I, the $m_{ij} = \alpha_0$ states are contained in class II, and the remaining states do not belong to either class I or class II. The latter states should be obtained at $\alpha = \alpha_0$ from nonlinear states which are not analytically known. More precisely, those states in (2) which are not in class I if $\alpha = \alpha_0$ (i.e., one or more of the m_{ij}'s equal to α_0) are not obtained from the states in (2) when $\alpha \to \alpha_0$ by superior value. On the contrary, when $\alpha \to \alpha_0$ by inferior value, there is a one to one mapping with the $\alpha = \alpha_0$ states. This failure to map exactly all states when $\alpha \to \alpha_0$ by superior value is pertinent only when α_0 is an even integer (Bose case). When α_0 is odd (Fermi case), one finds that the states which are not mapped either fall in class I or simply vanish, after proper symmetrization. Thus they can simply be ignored. This reflects the effect of the exclusion principle since the states which survive after symmetrization behave as $r_{ij}^{m_{ij}-\alpha_0}$ with $m_{ij} - \alpha_0 \geq 1$ when $r_{ij} \to 0$.

It follows that in order to control the gap above the N-anyon ground state (2), one should constrain the interval of definition of α. Starting from bosons, say $\alpha = 0$, the anyon eigenstates should be obtained by negative value $\alpha \leq 0$. Indeed, the ground states (2) interpolate continuously between the bosonic and fermionic ground states when α decreases from 0 to -1. Since the system is periodic in α with period 2, one might also consider the interval $[-2, -1]$. However, one knows that when $\alpha \to -2$ by superior value some unknown nonlinear states enter the game, as the gap between these peculiar states and the ground state decreases to 0 when $\alpha \to -2$. Clearly, in this region one cannot consider a consistent thermodynamic of the system in the ground state. Semiclassical and numerical analysis [6] for the few-anyon problem strengthen this analysis. In particular, if the semiclassical analysis indicates that the gap above the ground state is indeed of order $2\omega_c$ in the interval $\alpha \in [-1, 0]$, it shows that excited states merge in the ground state when $\alpha \to -2$. To conclude this discussion, the thermal probability to have an excited state is of order $\exp(-2\beta\omega_c)$ in the interval $\alpha \in [-1, 0]$. It is negligible when the thermal energy $1/\beta$ is smaller than the cyclotron gap. *The system is projected into the Hilbert space of the ground state.*

If one leaves aside the anyonic prefactor $\prod_{i<j} r_{ij}^{-\alpha}$, the N-anyon ground-state basis (2) can be rewritten as the direct product $\{\prod_{i=1}^{N} z_i^{\ell_i} \exp(-\frac{1}{2} m\omega_c z_i \bar{z}_i), \ell_i \geq 0\}$ of the one-body Landau ground states of energy ω_c and angular momentum ℓ_i. In this sense, the ground state of N anyons in a magnetic field ($L = \sum \ell_i$) is constructed in terms of one-body eigenstate in the lowest Landau level. As far as exact anyonic eigenstates are concerned, the m_{ij}'s basis has naturally prevailed. However, when symmetrizing the Fock space to derive the equation of state, the ℓ_i's basis will be definitively well adapted.

Since one knows exactly the N-anyon ground-state spectrum, one can compute the N-anyon partition function Z_N in the regime of strong magnetic field and low temperature. From the Z_N's one in principle deduces the cluster coefficients b_N. However, this algorithm happens to be quite tedious when N becomes large. Instead, one can choose to derive the equation of state in a second quantized formalism as a power series expansion in α. Both methods will be used below.

In order to study the statistical mechanics of an anyon gas, one should regularize the system at long distance to define a proper thermodynamic limit [7]. This is obviously still needed in the presence of the magnetic field. One confines the anyons by a harmonic attraction, adding $\sum_{i=1}^{N} \frac{1}{2} m\omega^2 r_i^2$ to the Hamiltonian (1). The thermodynamic limit is obtained when $\omega \to 0$. In the presence of the harmonic regulator, the ground-state problem is simply solved by replacing in (2) $\omega_c \to \omega_t = \sqrt{\omega^2 + \omega_c^2}$. The effect of the regulator is to partially lift the degeneracy of the ground-state spectrum

$$N\omega_c \to N\omega_t + \left\{ l + \sum_{i<j} (m_{ij} - \alpha) \right\} (\omega_t - \omega_c) \tag{3}$$

by $[L - \alpha N(N-1)/2](\omega_t - \omega_c)$, where $L - \alpha N(N-1)/2$ is interpreted as a total orbital angular momentum of the N-anyon ground state in the singular gauge. Again, if one leaves aside the anyonic prefactor, the eigenstates can be rewritten in terms of one-body harmonic Landau eigenstates

$$\varphi_{n,\ell}^{\omega}(z) = z^{\ell} L_n^{\ell}(m\omega_t z\bar{z}) \exp(-\tfrac{1}{2} m\omega_t z\bar{z}),$$
$$\epsilon_{n,\ell} = \omega_t(2n+1) + (\omega_t - \omega_c)\ell. \tag{4}$$

As already stressed above, N-anyon states have to be symmetrized in the case of boson based anyons. An N-anyon state is entirely characterized by the number n_ℓ of one-body Landau states of angular momentum $\ell = 0, 1, \ldots, \infty$, with the constraint $\sum_\ell n_\ell = N$, and its energy is nothing else but the sum of one-body harmonic Landau levels $\sum_\ell n_\ell \epsilon_{0,\ell}$ shifted by the constant $-N(N-1)\alpha(\omega_t - \omega_c)/2$. Since one is interested in the equation of state, symmetrization is done at the level of partition functions, in a way quite similar to the bosonic oscillator equation of state. The grand partition function for a gas of bosonic oscillators in the lowest Landau level is

$$Z^b = \prod_{\ell=0}^{\infty} \frac{1}{1 - z\exp(-\beta\epsilon_{0,\ell})}. \tag{5}$$

By definition, $Z^b = \sum_{N=0}^{\infty} z^N Z_N^b$, where z is the fugacity and Z_N^b is the N-boson partition function. For a linear spectrum one can use the identity $(1-ze^{-\beta\omega_t})Z^b \to Z^b$ when $z \to ze^{\beta(\omega_t-\omega_c)}$. One deduces that $Z_N^b - e^{-\beta\omega_t}Z_{N-1}^b = e^{-N\beta(\omega_t-\omega_c)}Z_N^b$ and finally

$$Z_N^b = \frac{e^{-N\beta\omega_t}}{(1-e^{-\beta(\omega_t-\omega_c)})(1-e^{-2\beta(\omega_t-\omega_c)})\cdots(1-e^{-N\beta(\omega_t-\omega_c)})}. \tag{6}$$

Since the anyonic interaction shifts the N-body ground-state spectrum by $-N(N-1)\alpha(\omega_t-\omega_c)/2$, the N-anyon partition function reads

$$Z_N = e^{\beta\frac{N(N-1)}{2}\alpha(\omega_t-\omega_c)}Z_N^b. \tag{7}$$

The thermodynamic limit, $\omega \to 0$, is understood as $1/(\beta\omega)^2 \to V/\lambda^2$, where the cluster coefficients b_N ($b_2 = Z_2 - \frac{1}{2}Z_1^2$, $b_3 = Z_3 - Z_2Z_1 + \frac{1}{3}Z_1^3, \ldots$) are multiplied by N accordingly [7–9]. One infers

$$b_N = \frac{V}{\lambda^2}2\beta\omega_c\frac{(N\alpha+1)(N\alpha+2)\cdots(N\alpha+N-1)}{N!} \times e^{-N\beta\omega_c}, \tag{8}$$

where V/λ^2 is the volume in units of the thermal wavelength $\lambda \equiv \sqrt{2\pi\beta/m}$.

One can see more directly how the volume factor, in the thermodynamic limit, materializes in the cluster coefficients using a second quantization language. One has to perform a perturbative expansion in α of the thermodynamical potential. In this context, short distance singularities of the anyon interaction $\frac{\alpha^2}{r_{ij}^2}$ should be treated before the perturbative analysis can proceed. These singularities manifest themselves in the nonanalyticity in $|\alpha|$ of the N-anyon spectrum, and the fact that N-anyon states have to vanish when two anyons approach each other. A perturbative analysis in α is possible [8] if the N-anyon wave function is rewritten as

$$\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \prod_{i<j} r_{ij}^{|\alpha|}\tilde{\psi}(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{9}$$

In (9), the exclusion of the diagonal of the configuration space, a nonperturbative effect in $|\alpha|$, has been encoded, by hand, in the N-anyon wave function ($\tilde{\psi}$ is assumed to be nonsingular). The Hamiltonian $\tilde{H}_N$ acting on $\tilde{\psi}$ is known to generate the correct perturbative expansion in α [8,9]. One notes that the redefinition (9) applied to the ground state (2) precisely factors out the anyonic prefactor $\prod_{i<j} r_{ij}^{-\alpha}$ in $\tilde{\psi}$. Thus, the $\tilde{\psi}$ ground-state basis does not depend on α, and is identical to the unperturbed basis. In the presence of the harmonic regulator $\tilde{H}_N^\omega$ reads

$$\tilde{H}_N^\omega = \sum_{i=1}^N\left[-\frac{2}{m}\partial_i\bar{\partial}_i + \frac{m}{2}\omega_t^2 z_i\bar{z}_i - \omega_c(z_i\partial_i - \bar{z}_i\bar{\partial}_i)\right] + \sum_{i<j}\left[-\frac{|\alpha|-\alpha}{m}\frac{\bar{\partial}_i-\bar{\partial}_j}{z_i-z_j} - \frac{|\alpha|+\alpha}{m}\frac{\partial_i-\partial_j}{\bar{z}_i-\bar{z}_j} + \alpha\omega_c\right]. \tag{10}$$

When acting on the ground-state basis it becomes a sum of one-body Hamiltonian $\sum_i -\frac{2}{m}\partial_i\bar{\partial}_i + \frac{m}{2}\omega_t^2 z_i\bar{z}_i - \omega_c(z_i\partial_i - \bar{z}_i\bar{\partial}_i)$ with total energy $\sum_i \epsilon_{0,\ell_i}$ shifted by $-\sum_{i<j}\alpha(\omega_t-\omega_c)$.

Second quantizing $\tilde{H}_N^\omega$ [9], the two-anyon vertex is simply the constant shift $-\alpha(\omega_t-\omega_c)/2$. One uses one particle Green's function in the lowest Landau level $G_\beta(z_2,z_1) \equiv \sum_{\ell=0}^\infty \varphi_{0,\ell}^\omega(z_2)\exp(-\beta\epsilon_{0,\ell})\bar{\varphi}_{0,\ell}^\omega(z_1)$ and computes the diagrammatic expansion of the thermodynamical potential $\Omega \equiv -\ln\sum_N Z_N z^N$ as a power series in α. At a given order α^n, the leading connected diagrams (which are the diagrams connected with $n+1$ loops) are indeed behaving as $1/(\beta\omega)^2$ when $\omega \to 0$. Also, at this order, nonvanishing diagrams start contributing in the cluster coefficient b_{n+1}. The thermodynamical potential is found to be

$$\Omega \equiv -\sum_{N=1}^\infty b_N z^N = -\frac{V}{\lambda^2}2\beta\omega_c\ln y(ze^{-\beta\omega_c}), \tag{11}$$

where $y(z')$ is a solution of $y - z'y^{\alpha+1} = 1$ with $y(z') \to 1$ when $z' \to 0$ [10]. The thermodynamical potential for bosons (fermions) is correctly reproduced when $\alpha = 0$ since $y = 1/(1-z')$ (respectively $\alpha = -1$ since $y = 1+z'$). The filling factor $\nu \equiv \rho/\rho_L$ (where $\rho_L = 2\beta\omega_c/\lambda^2$ is the Landau degeneracy per unit volume) as a function of z is given by $y(ze^{-\beta\omega_c}) = 1 + \nu/(1+\alpha\nu)$. It is monotonically increasing with z from 0 to $-1/\alpha$. One deduces the equation of state

$$P\beta = \rho_L\ln\left(1+\frac{\nu}{1+\alpha\nu}\right). \tag{12}$$

When expanding the pressure as a power series in the density ρ, one verifies that the expression of the second virial coefficient $a_2 = -\frac{1}{2\rho_L}(1+2\alpha)$ is reproduced in the limit where the Boltzmann weight $\exp(-2\beta\omega_c)$ is neglected. (One finds $a_N = (-\frac{1}{\rho_L})^{N-1}\frac{1}{N}[(1+\alpha)^N - \alpha^N]$.) Moreover, the first order expansion in α of (12) coincides with the perturbative result in a strong magnetic field given in [9]. The magnetization per unit volume is $\mathcal{M} = -\mu_0\rho + 2\frac{\mu_0}{\lambda^2}\ln(1+\frac{\nu}{1+\alpha\nu})$ where $\mu_0 \equiv |e|/2m$ is the Bohr magneton. Except near the singularity $\nu = -1/\alpha$, the ratio of the logarithmic correction to the de Haas–van Alphen magnetization $-\mu_0\rho$ is of order $(\beta\omega_c)^{-1}$, and thus negligible.

Both pressure and magnetization diverge at $\nu = -1/\alpha$. In the case $\alpha = 0$, any value of ν is allowed due to Bose condensation. In the case of Fermi statistics $\alpha = -1$, Pauli exclusion implies that the lowest Landau level is completely filled when $\nu = 1$. At a particular α, the critical value $\nu = -1/\alpha$ can be interpreted as at most

$-1/\alpha$ anyons of statistics α can occupy a given lowest Landau level. Since transitions to excited levels are by construction forbidden, the pressure necessarily diverges when the lowest Landau level is fully occupied such that any additional particle is excluded. In this situation the gas is incompressible. Indeed the isothermal compressibility coefficient $\chi_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,B}$ vanishes at the critical filling (except when $\alpha = 0$ where $\chi_T \to \lambda^2/2\omega_c$). The ground state is clearly nondegenerate as can be shown by extracting from Ω the canonical partition function of the critical system $Z_{\langle N\rangle_{\rm cr}} = \exp(-\beta\langle N\rangle_{\rm cr}\omega_c)$ where $\langle N\rangle_{\rm cr} = V\rho_L(-1/\alpha)$. Last but not least, one can get some information on the quantum numbers of the critical nondegenerate ground state. In the fermionic case $\alpha = -1$, the nondegenerate ground state is known to be a Vandermonde determinant, built from one-body Landau eigenstates $\ell_i = 0$ implying a minimal total angular momentum $\langle N\rangle_{\rm cr}(\langle N\rangle_{\rm cr} - 1)/2$ in the singular gauge. By analogy, in the case $\alpha \in [-1, 0]$ one infers that the state (2) with the ℓ_i's (or equivalently l and m_{ij}) all equal to 0,

$$\psi = \prod_{i<j} r_{ij}^{-\alpha} \exp\left(-\frac{m\omega_c}{2}\sum_i^N z_i\bar{z}_i\right), \qquad (13)$$

is the critical nondegenerate ground state with total angular momentum $-\alpha\langle N\rangle_{\rm cr}(\langle N\rangle_{\rm cr} - 1)/2$.

At the critical filling $\nu = -1/\alpha$, each anyon carrying $\alpha 2\pi/e$ individual flux, one gets that the flux of the magnetic field is precisely $-\langle N\rangle_{\rm cr}\alpha 2\pi/e$: the magnetic field is entirely screened by the anyons.

As already emphasized in the introduction, a similar magnetic screening is at the origin of the mean-field Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect [2]. If m is an odd integer, one can gauge transform the Hall electrons in boson based anyons carrying m quanta of flux, of opposite direction to the magnetic field. The mean-field solution is meaningful when the external magnetic field is completely screened by the flux tubes carried by the anyons. It precisely describes the $\nu = 1/m$ fractional Hall liquid. When $m = 1$, one has the $\nu = 1$ integer quantum Hall effect where the lowest Landau level is entirely filled. In the present work, where an exact solution has been obtained without relying on any mean-field approximation, each anyon carries $-\alpha$ quanta of flux, where $\alpha \in [-1, 0]$. At the critical filling $\nu = -1/\alpha$, again a magnetic screening occurs; it corresponds to the maximum filling of the lowest Landau level.

It would certainly be interesting to find out if something special happens in the particular case $\alpha = -1/n$ with n an integer ($n > 1$). At the critical filling $\nu = n$, the one-body Landau ground state has been filled exactly n times. So one has a nondegenerate ground state, with a cyclotron gap, and an integer filling n. This might suggest a possible reinterpretation of the n integer quantum Hall effect in terms of a critical anyon gas of statistics $-1/n$ in a strong magnetic field with Coulomb interactions ignored. In the usual picture of electrons filling n Landau levels, the external magnetic field is not screened. Here, on the contrary, the magnetic field is screened by the critical anyon gas, quite similarly to the Laughlin wave functions in the fractional quantum Hall effect.

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