Mesoscopic Dynamical Echo in Quantum Dots

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The time evolution of a wave packet within a disordered quantum dot is investigated. It is shown that although the disorder-averaged electron density is nearly homogeneous at times of the order of the diffusive time across the dot, at larger times there is a remarkable evolution towards a state with increasing correlation with the original state (i.e., an echo appears). At long times compared to the inverse mean level spacing the density distribution becomes time independent and preserves a memory of the original state. These effects are shown to reveal themselves in the relaxation of current through a quantum dot weakly coupled to reservoirs.

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In this Letter we discuss the dynamics of a quantum particle (electron) inside a quantum dot. Namely, we consider the time evolution of the probability to find the particle at a point $\mathbf{r}_2$ if it was located at a point $\mathbf{r}_1$ at time $t = 0$. We assume that the host potential of the dot contains a random component large enough for the mean free path of the particle within the dot, l, to be much smaller than the size of the dot, L. On the other hand, let l be much larger than the wavelength of the particle λ so that the dot is "metallic"; i.e., the relevant quantum states of the particles are extended within the dot.

In this case $\lambda \ll l \ll L$ we can distinguish between four different time regimes for the evolution of the wave packet.

(i) At the shortest times $t < \tau = l/v$ (v is the velocity of the particle) the evolution is ballistic and the size of the wave packet R increases linearly with time.

(ii) When time $t \geq \tau$ (or $R \geq l$) the particle dynamics becomes diffusive. The crossover from the ballistic to the diffusive regimes is well known in the theory of disordered metals. It was discussed, probably for the first time, in connection with the kinetic theory of gazes by Boltzmann who invented the "stosszahl ansatz" [1]. In the diffusive regime the size of the packet increases as $\sqrt{t}$ and the probability to find the particle at the original point is proportional to $t^{-d/2}$ where d is the dot dimensionality. Since the size of the quantum dot is finite this behavior lasts only until the size of the wave packet becomes comparable with L. This happens when $t \sim L^2/D = t_L$, where $D = vl/d$ is the diffusion constant.

(iii) Classical dynamics appears at $t > t_L$ and is not very interesting: The particle has already spread over all the system and the difference in the probabilities to find it at the original point and at any other point decreases exponentially with increasing t. In the quantum case the dynamics at the large times ($t > t_L$) is much less trivial and has not been previously discussed. This is the subject of our paper. We will see that the quantum interference and the discreteness of the exact energy levels of the particle in the dot lead to a nontrivial time dependence for $t \gg t_L$: The inhomogeneous part of the electron density distribution will increase linearly with time for $t < t_H = h/\Delta$ where Δ is the mean spacing between the energy levels.

(iv) The average electron density becomes time independent only for $t \gg t_H$. On the other hand this long-time limit distribution turns out to be essentially inhomogeneous in space and contains a memory about the original state. This memory can be eliminated only by inelastic scattering.

We now present the theory for the long-time behavior of a quantum particle inside a quantum dot. We will see that although all the calculations are made in the disordered case $l \ll L$ the results can be extended to the generic case of chaotic quantum motion within a finite volume. This theory enables us to describe also the ac conductivity and current relaxation through a dot very weakly coupled to external leads (see, e.g., Ref. [2]).

We start with the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + U_0(\mathbf{r}) + U_1(\mathbf{r}) \; ; \qquad (1)$$

$U_0(\mathbf{r})$ is the regular part of a confining potential while $U_1(\mathbf{r})$ represents a random potential. The problem can be solved for an arbitrary symmetry of the Hamiltonian. We will first describe the calculation in the unitary (broken T invariance) case and then present results for orthogonal and symplectic cases. To break T invariance we need a magnetic field which is represented in Eq. (1) by the vector potential $\mathbf{A}$. We will assume that the field is weak enough and neglect the Lorentz force [3].

The motion of the particle with energy ϵ_F from the original point $\mathbf{r}_1$ to a point $\mathbf{r}_2$ can be described by the conditional probability density to find the particle at a point $\mathbf{r}_2$ in a time t after it started at a point $\mathbf{r}_1$ at $t = 0$. This function can be expressed in terms of the exact re-

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tarded (advanced) Green function $G_{R(A)}$ of the Hamiltonian (1) (see, e.g., Refs. [4,5]):

$$P(t,\mathbf{r}) = \frac{1}{2\pi N(\epsilon_F)} \int \frac{d\omega}{2\pi} \exp(-i\omega t) \times \left\langle G_R\left(\epsilon_F + \frac{\omega}{2}; \mathbf{r}_1, \mathbf{r}_2\right) G_A\left(\epsilon_F - \frac{\omega}{2}; \mathbf{r}_2, \mathbf{r}_1\right)\right\rangle , \tag{2}$$

where $\langle \cdots \rangle$ stands for the averaging over $U_1(\mathbf{r})$, $N(\epsilon_F)$ is the density of states at energy ϵ_F, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The waiting function $p(t) \equiv P(t, \mathbf{r} = 0)$ is of special interest: $p(t)$ is the probability density for the particle to return at time t to the initial point $\mathbf{r}_1$.

$P(t,\mathbf{r})$ describes the ballistic spreading for $t < \tau$, the diffusive motion for $t > \tau$, and the transition between them. Following the usual procedure [6] we get from Eq. (2)

$$P(t,r) = \int \frac{d\omega}{2\pi} \exp(-i\omega t) \int \frac{d\mathbf{q}}{(2\pi)^d} \exp(-i\mathbf{q}\cdot\mathbf{r}) \times \left[I_d^{-1}(\omega + i/\tau, v_F|\mathbf{q}|) - 1/\tau\right]^{-1} , \tag{3}$$

where $d = 2, 3$ is the dimensionality of the sample ($d = 1$ has already been considered in Ref. [5] in detail), and

$$I_2(u,v) = i\left[u^2 - v^2\right]^{-1/2}, \quad I_3(u,v) = \frac{i}{v} \ln \frac{u+v}{u-v} . \tag{4}$$

Equations (3) and (4) lead to the well-known asymptotic results for the waiting functions

$$p(t \ll \tau) = \frac{1}{2\pi l} (v_F t)^{1-d} , \tag{5}$$

$$p(t \gg \tau) = (4\pi D t)^{-d/2} . \tag{6}$$

Equation (5) describes the returning to the original point after a single scattering event, while Eq. (6) is a result of many scatterings by randomly distributed centers.

Equation (6) is valid only if the diffusion length $R(t) = \sqrt{2dDt}$ does not exceed the size of the quantum dot L, i.e., when $t < t_L \sim 1/E_c$, where $E_c = 2dD/L^2$ is the Thouless energy. In the opposite limit $t > 1/E_c$, the time evolution is determined by boundary scattering. Formally this shows up in the discreteness of the spectrum of the diffusion equation with mean energy separation E_c. Since this spectrum starts from zero [7,8], $P(t,\mathbf{r})$ at $t > 1/E_c$ can be written as

$$P(t,\mathbf{r}) = (1/V)\left[1 + O(\exp(-tE_c))\right] , \tag{7}$$

where $V \sim L^d$ is the volume of the dot. Equation (7) means that the deviation of the density distribution from the homogeneous one is exponentially small. Equation (7) can be rewritten in the ω representation in the form

$$\tilde{P}(\omega,\mathbf{r}) = \frac{1}{V} \frac{1}{-i(\omega + i0)} \left[1 + O\left(\frac{\omega}{E_c}\right)\right] . \tag{8}$$

We are now going to demonstrate that Eq. (7) describes only an intermediate rather than the final stage of the overall relaxation process. A nontrivial behavior of the function $P(t,r)$ for $t > E_c^{-1}$ appears thanks to the quantum (so-called weak localization, see, e.g., Refs. [3,9]) corrections to the diffusion: Quantum interference leads to corrections to Eq. (8) in addition to the expansion in ω/E_c. These corrections can be expressed in terms of an expansion in $1/\alpha$, where α is the new quantum parameter of the present problem

$$\alpha = -i\pi(\omega + i0)/\Delta , \quad \Delta = [N(\epsilon_F)V]^{-1} . \tag{9}$$

Here Δ is the mean spacing between exact energy levels of the particle in the quantum dot. The dimensionless ratio E_c/Δ is known to be equal to the conductance g of the sample, $g = E_c/\Delta = DN(\epsilon_F)V/L^2$. As we have already mentioned we are considering the case of the weak disorder so that $g \gg 1$.

Equation (8) is valid provided $1 \ll \alpha \ll g$, i.e., in the frequency interval $\Delta \ll \omega \ll E_c$. We will see later that $P(t,\mathbf{r})$ at $t > 1/\Delta$ differs substantially from Eq. (7). At the same time it turns out to be impossible to evaluate its Fourier transform $\tilde{P}(\omega,\mathbf{r})$ within a perturbation theory in $1/\alpha$, for there is an essential singularity at $1/\alpha = 0$. Fortunately it is possible to determine $\tilde{P}(\omega,\mathbf{r})$ exactly in the whole region $\alpha \ll g$, including $\alpha \ll 1$ and $\alpha \gg 1$.

The method which enables us to comprehensively study disorder in a closed sample in the frequence region $\omega \ll E_c$ is the supersymmetry σ model invented in Ref. [7]. In terms of this method the correlation function $\tilde{P}(\omega,\mathbf{r})$ can be written in the form

$$\tilde{P}(\omega,\mathbf{r}) = (1/2)\pi N(\epsilon_F)\left[a(\omega) + k_d(\mathbf{r})b(\omega)\right] . \tag{10}$$

Here

$$a(\omega) = \langle Q_{33}^{12}\, Q_{33}^{21}\rangle_Q , \quad b(\omega) = \langle Q_{33}^{11}\, Q_{33}^{22}\rangle_Q , \tag{11}$$

where $\langle \cdots \rangle_Q$ means a functional averaging over the supermatrix fields Q with the weight $\exp(-F)$, where F is the free energy functional of the σ model. In Eq. (10) we use the same notations for the supermatrix elements as in Ref. [7]. At $\omega \ll E_c$ the standard σ model can be reduced to its zero-dimensional version where functional integrals over Q become definite ones. This enables us to derive Eq. (10) for arbitrary large times.

The coordinate dependence in Eq. (10) is determined by $k_d(r)$ which equals

$$k_d(\mathbf{r}) = \langle G(\mathbf{r})\rangle^2 \langle G(0)\rangle^{-2} , \tag{12}$$

where $G(\mathbf{r})$ is the Green function of the equation

$$(\epsilon_F - \hat{H})G(\mathbf{r}) = \delta(\mathbf{r}) . \tag{13}$$

Equations (12) and (13) mean that $k_d(\mathbf{r})$ is of a ballistic nature. For $d = 3$ for a flat background potential with weak disorder we find

$$k_3(\mathbf{r}) = \frac{\sin^2(k_F|\mathbf{r}|)}{(k_F|\mathbf{r}|)^2} \exp\left(-\frac{|\mathbf{r}|}{l}\right) . \tag{14}$$

Similarly for $d = 2$ the function $k_2(\mathbf{r})$ is given by the equation

$$\sqrt{k_2(\mathbf{r})} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dy}{y^2+1} J_0\left(k_F|\mathbf{r}| + \frac{|\mathbf{r}|}{2l} y\right) ,$$

where $J_0(y)$ is the Bessel function. When $k_F|\mathbf{r}| \gg 1$ we have then

$$k_2(\mathbf{r}) = \frac{4}{\pi k_F|\mathbf{r}|} \cos^2\left(k_F|\mathbf{r}| - \frac{\pi}{4}\right) \exp\left(-\frac{|\mathbf{r}|}{l}\right) , \qquad (15)$$

and $k_2(\mathbf{r}) = 1$ in the region $k_F|\mathbf{r}| \ll 1$.

In the limit $\omega \ll E_c$ the energy functional of the σ model F takes the form

$$F = -(\alpha/4) \, \mathrm{Str}(\Lambda Q) , \qquad (16)$$

where $\Lambda^{11} = -\Lambda^{22} = 1$, and $Q^2 = 1$ with Str being the supertrace. Averaging in Eq. (11) with F from Eq. (16) can be carried out by using the parametrization of Ref. [7] and we obtain for the unitary ensemble

$$a(\omega) = 2/\alpha , \quad b(\omega) = 1 + (1 - e^{-2\alpha})/\alpha^2 . \qquad (17)$$

Equation (17) is valid for arbitrary $\alpha \ll g$.

One can see that the expansion of $b(\omega)$ in $1/\alpha$ contains only terms of order $1/\alpha^2$. On the other hand, $b(\omega)$ also contains a nonanalytical part $\propto (1/\alpha^2)\exp(-2\alpha)$.

Equation (10) enables us to write $P(t,\mathbf{r})$ in the form

$$P(t,\mathbf{r}) = [1 + k_d(\mathbf{r}) C(z)] \; / \; V , \qquad (18)$$

where $z = t\Delta/2\pi$ and in the unitary case $C(z)$ is

$$C_{\mathrm{unit}}(z) = z\theta(1-z) + \theta(z-1) . \qquad (19)$$

Here $\theta(z)$ is the step function. The present universal time dependence (19) is directly related to the property of the electron spectrum, its rigidity, in the mesoscopic sample and, therefore, the echo described by Eq. (18) differs from the ringing effect considered in Ref. [10].

According to Eqs. (18) and (19), for times longer than $2\pi/\Delta$ the following stationary distribution is established:

$$p_\infty(\mathbf{r}) \equiv \lim_{t\to\infty} P(t,\mathbf{r}) = [1 + k_d(\mathbf{r})] \; / \; V . \qquad (20)$$

Therefore the memory about the initial position is preserved forever (provided there is no phase breaking). In Fig. 1 the electron density distribution averaged over the atomic scale is schematically shown for different stages of the time evolution.

It is noteworthy that $p_\infty(\mathbf{r})$ can be expressed through the exact wave functions of the particle $\psi_\nu(\mathbf{r})$ in the dot

$$p_\infty(\mathbf{r}) = \sum_\nu \langle |\psi_\nu(\mathbf{r}_1)|^2 \, |\psi_\nu(\mathbf{r}_2)|^2 \, \delta(\epsilon_F - \epsilon_\nu) \rangle , \qquad (21)$$

and thus the function $p_\infty(\mathbf{r})$ describes the space correlation of the wave functions.

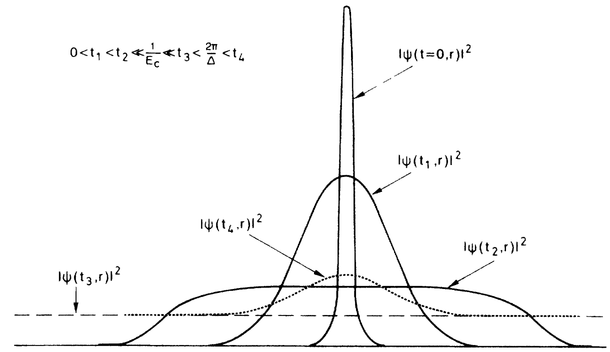

FIG. 1. Schematic view of a time evolution of electron wave packet in a mesoscopic unitary sample: $|\psi(t,\mathbf{r})|^2$ is the electron density averaged over atomic scale.

The result for $C(z)$ in (18) for the orthogonal ensemble is

$$C_{\mathrm{ort}}(z) = 1 + z[2 - \ln(1+2z)]\theta(1-z) + \left[2 - z\ln\frac{1+2z}{2z-1}\right]\theta(z-1) . \qquad (22)$$

The first term in Eq. (22) is due to the constructive interference of backscattering: This "Cooperon" contribution already exists starting at time τ; see, e.g., Ref. [11]. Therefore, in the present case the inhomogeneous distribution remains over times $t \gtrsim 1/E_c$ and its amplitude additionally increases for times $t \sim 2\pi/\Delta$. As a result the inhomogeneous component of the long-time limit distribution $p_\infty(\mathbf{r})$ is twice as large as in the unitary case.

For the symplectic ensemble $C(z)$ in Eq. (18) is

$$C_{\mathrm{sympl}}(z) = -1 + z[2 - \ln|1-2z|]\theta(1-z) + 2\theta(z-1) . \qquad (23)$$

According to Eqs. (18) and (23), at times $t \sim \pi/\Delta$ an appreciable part of the electron density turns out to be again concentrated around the original point. Equation (23) fails only in the exponentially short interval $|z - 1/2| \ll \exp[-(L/l)(Lk_F)^{d-1}]$. This strong interference is due to the Kramers degeneracy of the spectrum [12]. In fact, in all three cases $C(z)$ is the Fourier transform of the two-level correlation function (see, e.g., Ref. [7]).

A physical dynamical quantity that can be measured experimentally is a transient current through the dot [10,13]. The point contacts between the dot and the external leads can be well described by the transmission coefficients $T_{1,2}$. For far separated contacts one can find, in the linear approximation in the abruptly applied driving voltage and using the supersymmetry formalism [13], that the current as a function of time is

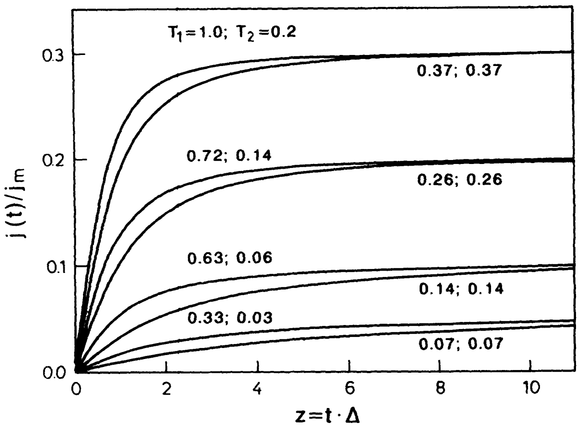

FIG. 2. Transient current through a quantum dot as a function of time for different transmission coefficients $T_{1,2}$ between the dot and external leads.

$$\frac{j(t)}{j_m} = \frac{2T_1T_2}{(T_1-T_2)^2}\left\{\frac{T_1T_2(2-T_1-T_2)}{T_2-T_1}\ln\frac{1+zT_1}{1+zT_2}\right.$$
$$+ (T_1+T_2-2T_1T_2)\,[z-(z-1)\theta(z-1)]$$
$$-(1-T_1)\,\phi_2(z)-(1-T_2)\,\phi_1(z)$$
$$-\frac{T_1^2(1-T_2)+T_2^2(1-T_1)}{T_1-T_2}\left[\left(\frac{1}{T_1}+z-1\right)\phi_1(z)\right.$$
$$\left.\left.-\left(\frac{1}{T_2}+z-1\right)\phi_2(z)\right]\right\}, \qquad (24)$$

where $z = t\Delta/2\pi$, $j_m = (2e^2/h)V_{12}$, with V_{12} the applied voltage, and

$$\phi_{1,2}(z) = \ln[1+zT_{1,2}] - \ln[1+(z-1)T_{1,2}]\theta(z-1)\ . \quad (25)$$

Above, we restricted the consideration to the unitary case and the time interval $t \gg 1/E_c$.

As is seen from Eq. (24), $j(t)/j_m$ is a universal function of time measured in units of the inverse energy spacing Δ and the transmission coefficients. At first, for $1/g \ll z \ll 1$, the current $j(t)/j_m$ rises linearly with time. This result can be easily understood: In the frequency region $T_{1,2}\Delta \ll \omega \ll E_c$ the system is inertial and its conductance should be purely inductive. In accordance with (8) and Ref. [12] the conductance is approximately equal to

$$G(\omega) = \left(\frac{2e^2}{h}\right)\frac{2T_1T_2}{\pi}\frac{\Delta}{-i(\omega+i0)}\ . \qquad (26)$$

At the times $t \gtrsim 2\pi/\Delta$ the linear increase of the current with time slows down because of just the above return processes and due to the leakage into the external leads. Simultaneously the phase-breaking processes at the contacts start to play a role. Finally for the times $t \gtrsim 2\pi/T_{1,2}\Delta$ the current reaches a steady state value determined by the static conductance

$$G(0) = \frac{2e^2}{h}\left\{\frac{2T_1T_2(T_1+T_2-2T_1T_2)}{(T_1-T_2)^2} - \frac{2(T_1T_2)^2(2-T_1-T_2)}{(T_1-T_2)^3}\ln\frac{T_1}{T_2}\right\}\ .$$

In Fig. 2 we show examples of the time dependence of the current for a few values of the transmissions coefficients. The echo effect becomes explicitly visible as a break point at $t = 2\pi/\Delta$ in $j''(t)$, which corresponds to a response to the two short consecutive rectangular voltage pulses of opposite polarizations.

In conclusion, we have studied the internal spatial structure of the wave functions in the quantum dot, where the electronic motion is governed by chaotic dynamics. We have found that the spatial correlations decay in accordance with the laws of ballistic spreading. The final mesoscopic stage of time evolution of the electron within the dot follows a universal time dependence determined by the inverse Fourier transform of the two-level correlation function.

Experimentally the mesoscopic dynamics could be observed through the time dependence of transient currents through the dot, which is shown to be a universal two-parameter function of time with the parameters being the transmission coefficients of the contacts.

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[1] L. Boltzmann, Ber. Wein. Akad. **66**, 275 (1872).
[2] M.A. Kastner, Rev. Mod. Phys. **64**, 849 (1992).
[3] P. Lee and T.R. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[4] E.N. Economou and M.H. Cohen, Phys. Rev. B **5**, 2931 (1972).
[5] E.P. Nakhmedov, V.N. Prigodin, and Yu.A. Firsov, Zh. Eksp. Teor. Fiz. **92**, 2133 (1987) [Sov. Phys. JETP **65**, 1202 (1987)].
[6] A.A. Abrikosov, L.P. Gor'kov, and I.E. Dzyaloshinskii, *Method of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, NJ, 1963), Chap. 3.
[7] K.B. Efetov, Adv. Phys. **32**, 53 (1983).
[8] B.L. Altshuler and B.I. Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[9] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.R. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[10] N.S. Wingreen, A-P. Jauho, and Y. Meir, Phys. Rev. B **48**, 8487 (1993).
[11] G. Bergmann, Phys. Rep. **107**, 1 (1984).
[12] H.A. Kramers, Proc. Acad. Sci. Amsterdam **33**, 959 (1930).
[13] V.N. Prigodin, K.B. Efetov, and S. Iida, Phys. Rev. Lett. **71**, 1230 (1993).