Effective Critical Exponents for Dimensional Crossover and Quantum Systems from an Environmentally Friendly Renormalization Group

Denjoe O'Connor
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

C. R. Stephens
Institute for Theoretical Physics, Rijksuniversiteit Utrecht, Princetonplein 5, 3508TA Utrecht, Netherlands
(Received 30 August 1993)

Series for the Wilson functions of an "environmentally friendly" renormalization group are computed to two loops, for an O(N) vector model, in terms of the "floating coupling," and resummed by the Padé method to yield crossover exponents for finite size and quantum systems. The resulting effective exponents obey all scaling laws, including hyperscaling in terms of an effective dimensionality, $d_{\rm eff}=4-\gamma_\lambda$, which represents the crossover in the leading irrelevant operator, and are in excellent agreement with known results.

PACS numbers: 64.60.Ak, 05.70.Fh, 11.10.Gh, 61.41.+e

Physical systems can exhibit different scaling behavior in different asymptotic regimes. The crossover between such asymptotic regimes is important both theoretically and experimentally. One may think of a crossover as being induced by some "environmental" variable. Two of the most interesting crossovers are those induced by finite-size [1] and quantum [2] effects. For finite-size systems the environmental variable is L, the system size, while in quantum systems it is the inverse absolute temperature, $\beta\hbar$ [3].

The main difficulty in treating systems which exhibit a crossover is that the qualitative nature of the effective degrees of freedom (DOF), i.e., the fluctuations, changes significantly as a function of scale, being very sensitive to the environment in the crossover region. The renormalization group (RG) is our most powerful tool for investigating such changes. If one views the RG transformation as a "coarse graining" procedure, one must ask whether a particular coarse graining captures the qualitative changes associated with the crossover. One must be careful not to throw away important environment dependence by arguing for g independent RG flow equations from the limit of large momenta $k\gg g$, where g is the characteristic scale set by the environment, as when propagated to scales $k<g$ these coarse grain effective DOF which are a very poor representation of the system's fluctuations at that scale. We call an RG which tracks the changing nature of the effective DOF—"environmentally friendly."

The most accurate RG results for properties of critical systems have been achieved by applying field theoretic techniques [4], and are in very impressive agreement with experiment [5]. Three approaches have been used: ε expansion [6], N^{-1} expansion [7], and fixed dimension perturbation theory [8]. The first fails in crossovers where the upper critical dimension changes, such as in finite-size crossover. The second fails in crossovers where the order parameter can change its symmetry, such as bicritical crossover. We adopt the spirit of the fixed dimension approach, if not the letter, in the context of environmentally friendly renormalization.

Field theory RGs, historically, have emphasized the role of ultraviolet divergences, which are independent of infrared scales and therefore environment insensitive; however, failure to track the changing nature of the effective DOF leads, typically, to a breakdown of perturbation theory. More environmentally friendly RGs have been implemented in some contexts [9–13]. Schmeltzer [14] calculated $\gamma_{\rm eff}$ to one loop for a three-dimensional quantum ferroelectric, and Lawrie [15] considered dimensional crossover for d-dimensional quantal and $(d+1)$-dimensional finite-sized Ising models for $3<d<4$ using an ε expansion. Unlike our method the ε expansion could not capture the crossover between two nontrivial fixed points as the upper critical dimension changes across the crossover. Field theoretic results for dimensional crossover in a fully finite geometry or a cylinder have been obtained [16] but the techiques used have not been extended to the case of a system with more than one fixed point.

Though our general approach is applicable to a very wide class of crossovers [11,17], we restrict our attention to finite-size crossover and quantum-classical crossover. We begin with the "microscopic" Landau-Ginzburg-Wilson Hamiltonian

$$H[\varphi_B]=\int_0^L\int d^{d+1}x\left[\frac{1}{2}(\nabla\varphi_B)^2+\frac{1}{2}m_B^2\varphi_B^2+\frac{1}{2}t_B(x)\varphi_B^2+\frac{\lambda_B}{4!}\varphi_B^4-H_B(x)\varphi_B\right], \quad (1)$$

which describes either a layered $(d+1)$-dimensional system of thickness L or a d-dimensional quantum system with $L=\beta\hbar$, β being the inverse temperature. We will assume the order parameter possesses an O(N) symmetry; the case $N=1$ of quantum-classical crossover represents an Ising model in a transverse magnetic field. In the finite-size case $m_B^2+t_B=T-T_0$, and in the Ising model in a transverse field $m_B^2+t_B=\Gamma-\Gamma_0$. Here T_0 and Γ_0 are the critical temperature and transverse field, respectively, in the mean field approximation.

0031-9007/94/72(4)/506(4)$06.00

L-dependent normalization conditions yield an environmentally friendly RG and ensure that all diagrammatic information is exponentiated in the solution of the resulting RG equation. The relation between the bare and renormalized vertex functions is $\Gamma_B^{(N,M)} = Z_\phi^{-N/2} Z_{\phi^2}^{-M} \times \Gamma^{(N,M)}$. The renormalized dimensionful coupling is similarly related to the bare one by $\lambda_B = Z_\lambda^{-1}\lambda$ (for notation see [4]). We choose the normalization conditions

$$\begin{aligned}
&\text{(i)}\ \Gamma^{(2)}(k=0,t=\kappa^2,\lambda,L,\kappa)=\kappa^2 ,\\
&\text{(ii)}\ \frac{\partial\Gamma^{(2)}}{\partial k^2}(k,t=\kappa^2,\lambda,L,\kappa)\Big|_{k=0}=1 ,\\
&\text{(iii)}\ \Gamma^{(4)}(k=0,t=\kappa^2,\lambda,L,\kappa)=\lambda ,\\
&\text{(iv)}\ \Gamma^{(2,1)}(k=0,t=\kappa^2,\lambda,L,\kappa)=1 ,
\end{aligned} \tag{2}$$

which specify Z_ϕ, Z_{ϕ^2}, and Z_λ. Condition (i), together with the multiplicative renormalization of t, implies that t is proportional to $T-T_c(L)$ for the finite-size system, and $\Gamma-\Gamma_c(\beta)$ for the quantal Ising model, i.e., that one is measuring temperature/field deviations relative to the L-dependent critical point. We are assuming the system exhibits critical behavior for any value of L, which restricts our attention to $d>1$ in the case of $N>1$, but in no way restricts the generality of our approach [18]. The RG equation derives from the normalization point independence of the bare theory, i.e., $\kappa d/d\kappa\Gamma_B^{(N)}=0$. Using the relation between the bare and renormalized vertex functions and expressing things in terms of the renormalized parameters the infinitesimal form of the RG equation then becomes

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial\lambda}+\gamma_{\phi^2}t\frac{\partial}{\partial t}-\frac{1}{2}\gamma_\phi\left[N+\bar\phi_B\frac{\partial}{\partial\bar\phi_B}\right]\right]\Gamma^{(N)}=0 \tag{3}$$

with

$$\gamma_\phi=\frac{1}{Z_\phi}\kappa\frac{dZ_\phi}{d\kappa},\quad \gamma_{\phi^2}=\frac{1}{Z_{\phi^2}^{-1}}\kappa\frac{dZ_{\phi^2}^{-1}}{d\kappa},\quad \frac{\beta(\lambda)}{\lambda}=\gamma_\lambda=\frac{1}{Z_\lambda}\kappa\frac{dZ_\lambda}{d\kappa}. \tag{4}$$

The functions γ_ϕ, γ_{ϕ^2}, and γ_λ are the Wilson functions. They are explicitly L dependent due to the normalization conditions (2) and all the physics of the crossover can be gleaned from them. Note that we are here using an RG which runs the renormalized temperature parameter in distinction to the Callan-Symanzik equation which runs the physical correlation length.

A suitable coupling, with respect to which perturbation theory can be performed, is the floating coupling [11,19], h, which is chosen so as to make the quadratic term in $\beta(h)$ have unit coefficient. Our perturbation theory is then carried out at the level of the Wilson functions in terms of h. The expressions obtained are, however, only the leading terms in an asymptotic expansion of the functions $\beta(h,z)$, $\gamma_{\phi^2}(h,z)$, and $\gamma_\phi(h,z)$. We use [2/1] Padé approximants to resum these asymptotic series obtaining

$$\beta(h,z)=-\varepsilon(z)h+\frac{h^2}{1+4\{[(5N+22)/(N+8)^2]f_1(z)-[(N+2)/(N+8)^2]f_2(z)\}h}, \tag{5}$$

$$\gamma_\phi(h,z)=2\frac{(N+2)}{(N+8)^2}f_2(z)h^2, \tag{6}$$

$$\gamma_{\phi^2}(h,z)=\frac{(N+2)}{(N+8)}\frac{h}{1+[6/(N+8)][f_1(z)-\frac{1}{3}f_2(z)]h}, \tag{7}$$

where the functions ε, f_1, and f_2 depend on d and $z=\kappa L$ but are independent of N. The original non-Padé resummed series can be recovered by expanding $1/(1+xh)\sim 1-xh$. We will take the solution of (5) as our perturbation parameter. After these equations are solved it is then inappropriate to do any further expansion.

The functions $\varepsilon(z)$, $f_1(z)$, and $f_2(z)$ are the basic building blocks, their specific functional form depending on the particular crossover in question. $\varepsilon(z)$ can be thought of as being a measure of the "effective dimensionality" of the system. Numerical evidence for an effective dimensionality was found in [20]. The functions f_1 and f_2 for general d and the crossovers of interest here can be found in [17]. For $d=3$, the expressions become especially simple; we find

$$\varepsilon(z)=1-z\frac{d}{dz}\ln\left[\sum_n m^{-3}\right],$$

$$f_1(z)=2\frac{\sum_{n_1,n_2}[(1/m_1^3)(1/M-1/2m_2)+(1/m_1M^2)(1/m_1+2/m_2)]}{(\sum_n 1/m^3)^2},$$

$$f_2(z)=4\frac{\sum_{n_1,n_2}1/M^3m_1}{(\sum_n 1/m^3)^2},$$

with $m_i=(1+4\pi^2n_i^2/z^2)^{1/2}$, $m_{12}=[1+(4\pi^2/z^2)(n_1+n_2)^2]^{1/2}$, $M=m_1+m_2+m_{12}$. We plot $\varepsilon(z)$, $f_1(z)$, and $f_2(z)$ against $\ln(1/z)$ in Fig. 1.

Effective critical exponents, defined as logarithmic derivatives of the associated thermodynamic quantities with respect to $T-T_c(L)$ at fixed L, for the finite-size crossover and with $\Gamma-\Gamma_c(\beta)$ for fixed β in the quantum problem, using the above RG [11,21] can be shown to obey all the usual scaling relations including hyperscaling. The usual dimen-

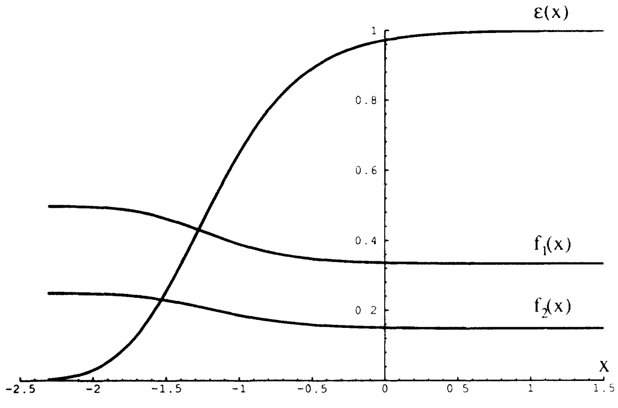

FIG. 1. $\varepsilon(x), f_1(x)$, and $f_2(x)$ vs $x=\ln(\xi_L/L)$.

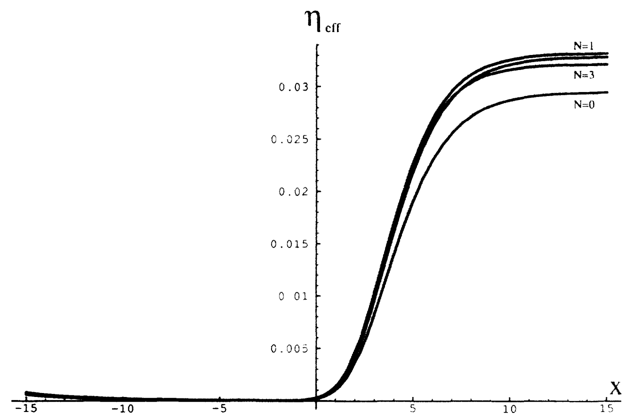

FIG. 3. $\eta_{\rm eff}$ vs $x=\ln(\xi_L/L)$ for $N=0,1,2$, and 3.

sion is replaced by the effective dimension $d_{\rm eff}=4-\gamma_\lambda$ which reflects the changing importance of the leading irrelevant operator. As a consequence these exponents are related to the Wilson functions through $\nu_{\rm eff}=1/(2-\gamma_{\phi^2})$, $\eta_{\rm eff}=\gamma_\phi$, $\gamma_{\rm eff}=(2-\gamma_\phi)/(2-\gamma_{\phi^2})$, $\alpha_{\rm eff}=(\gamma_\lambda-2\gamma_{\phi^2})/(2-\gamma_{\phi^2})$, $\beta_{\rm eff}=(2-\gamma_\lambda+\gamma_\phi)/(4-2\gamma_{\phi^2})$, and $\delta_{\rm eff}=(6-\gamma_\lambda-\gamma_\phi)/(2-\gamma_\lambda+\gamma_\phi)$. Analogous effective exponents associated with variations with respect to L at fixed T, and β at fixed Γ, can also be defined and computed.

We present our results in Figs. 2-5. In all graphs the horizontal axis is $\ln(\xi_L/L)$. The different curves correspond to $N=0$ (polymers), $N=1$ (Ising), $N=2$ (*XY*), $N=3$ (Heisenberg), and $N=\infty$ (spherical model) and represent both a four-dimensional layered geometry of thickness L and a three-dimensional quantum model at $\beta\hbar=L$. The logarithmic corrections to scaling at the bulk end are clearly visible, and are as expected from four-dimensional calculations. All curves are with the boundary condition $h=1$ at $\ln(\xi_L/L)=-20$; the value of h at the initial scale parametrizes different possible crossover curves but all curves asymptote to the same form. In Fig. 2 we plot $\nu_{\rm eff}$, the correlation length exponent, for $N=\infty$, $\nu_{\rm eff}\equiv 1/(d_{\rm eff}-2)$ across the entire crossover. In Fig. 3 we plot $\eta_{\rm eff}$, the exponent which governs the falloff in critical correlations. This exponent is not a monotonic function of N; it is identically zero for $N=-2$ and $N=\infty$ and attains a maximum for some intermediate value. This is the least accurate of our exponents and the peak appears to be at $N=1$, though more accurate values for this exponent suggest it occurs at higher values, probably $N=3$. Figure 4 shows a plot of the effective specific heat exponent $\alpha_{\rm eff}$ which measures how the singular part of the free energy changes as Γ or T varies. The extra case $N=-2$ is added here, since, in the case of dimensional crossover it is distinguishable from the Gaussian model due to the fact that γ_λ for the latter is zero, whereas for the former it is nonzero, being a measure of the changing effect of the leading irrelevant operator. Across the entire crossover one has $\alpha_{\rm eff}=2-\nu_{\rm eff}d_{\rm eff}$. Not only does one see the change in sign of the specific heat exponent as a function of N, but also one sees that the effective specific heat exponent can change sign as a function of ξ_L/L. This is quite pronounced for the *XY* model which starts off positive, increases, then turns negative at

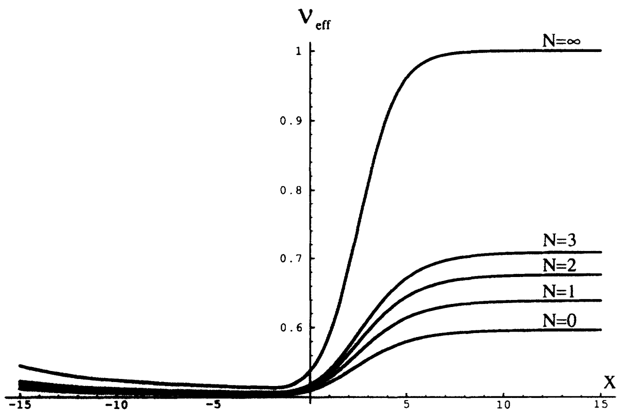

FIG. 2. $\nu_{\rm eff}$ vs $x=\ln(\xi_L/L)$ for $N=0$, 1, 2, 3, and ∞.

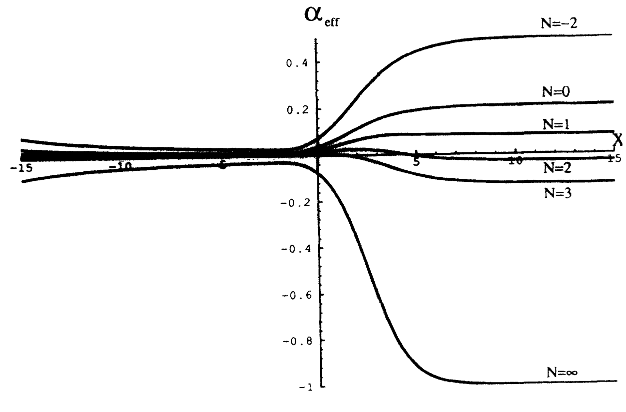

FIG. 4. $\alpha_{\rm eff}$ vs $x=\ln(\xi_L/L)$ for $N=-2$, 0, 1, 2, 3, and ∞.

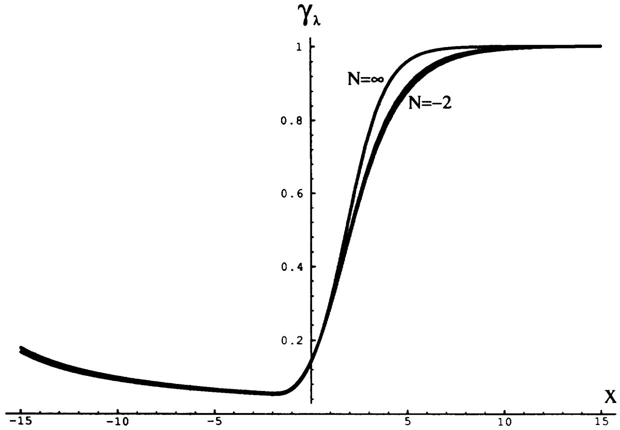

FIG. 5. γ_λ vs $x = \ln(\xi_L/L)$ for $N = -2$, 0, 1, 2, 3, and ∞.

$\xi_L \sim 100L$. It would be interesting, based on the Harris criterion for the relevance or irrelevance of weak disorder, to see whether disorder could change from being irrelevant to relevant as a function of size, or temperature in the case of a quantum system. In Fig. 5 we plot $\gamma_\lambda = 4 - d_{\rm eff}$ which also gives information about the effective dimensionality of the system. Notice that γ_λ is very robust to changes in N, varying very little across the entire range of N, $[-2, \infty]$. The other effective exponents can be determined from the effective exponent laws, which we have verified also by direct calculation. Asymptotic values of critical exponents and associated quantities are tabulated below (see Table I). All these values are in very good agreement with corresponding high temperature series and experimental results (see [22] and references therein). We believe the entire crossover curves are of similar accuracy.

In summary, two loop Padé resummed perturbation theory for an environmentally friendly RG yielded effective exponents for dimensional crossover in a four-dimensional layered system with periodic boundary conditions and for quantum to classical crossover in three dimensions. We paid special attention to polymers, the Ising, XY, Heisenberg, and spherical models. Asymptotic values for the exponents of these systems are in very good agreement with known results and experiment. Our general formalism is applicable to a wide class of crossovers. Higher loop calculations should yield effective exponents to the precision of standard critical exponents [23]. There is merit in pursuing such calculations as our methods provide a direct and physical connection between exponents in different dimensions.

TABLE I. Asymptotic critical exponents.

N	γ_ϕ	γ_{ϕ^2}	h	$\gamma_{\rm eff}$	$\nu_{\rm eff}$	$\alpha_{\rm eff}$
−2	0[a]	0[a]	1.800	1[a]	0.5[a]	0.5[a]
−1	0.0200	0.145	1.820	1.088	0.550	0.351
0	0.0295	0.277	1.785	1.175	0.596	0.211
1	0.0329	0.388	1.732	1.257	0.639	0.083
2	0.0332	0.479	1.675	1.330	0.676	−0.029
3	0.0322	0.552	1.621	1.395	0.709	−0.126
4	0.0305	0.611	1.573	1.451	0.737	−0.211
∞	0[a]	1[a]	1[a]	2[a]	1[a]	−1[a]

[a]These values are exact.

[1] M. N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983), Vol. 8; *Finite-Size Scaling*, edited by J. L. Cardy, Current Physics Sources and Comments Vol. 2 (North-Holland, Amsterdam, 1988).
[2] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976); S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B **39**, 2344 (1989).
[3] Other environmental factors that may induce a crossover are long range interactions, anisotropic interactions, external fields, boundary conditions, etc.
[4] E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. Green (Academic, New York, 1976), Vol. 6.
[5] G. A. Baker, B. G. Nickel, and D. I. Meiron, Phys. Rev. B **17**, 1365 (1978).
[6] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 248 (1972).
[7] R. Abe, Prog. Theor. Phys. **48**, 1414 (1972).
[8] G. Parisi, J. Stat. Phys. **23**, 49 (1980).
[9] D. J. Amit and Y. Y. Goldschmidt, Ann. Phys. (N.Y.) **114**, 356 (1978).
[10] D. R. Nelson and E. Domany, Phys. Rev. B **13**, 236 (1976); P. Seglar and M. E. Fisher, J. Phys. C **13**, 6613 (1980).
[11] Denjoe O'Connor and C. R. Stephens, Proc. R. Soc. London A (to be published).
[12] E. Frey and F. Schwabl, Phys. Rev. B **42**, 8261 (1990); C. R. Stephens, J. Magn. Magn. Mater. **104-107**, 297 (1992).
[13] For a bicritical crossover we find [11] the characteristic "dip," in agreement with [10] but not with [9].
[14] D. Schmeltzer, Phys. Rev. B **32**, 7512 (1985).
[15] I. D. Lawrie, J. Phys. C. **11**, 3857 (1978).
[16] E. Brézin and J. Zinn-Justin, Nucl. Phys. **B257 [FS14]**, 867 (1985); J. Rudnick, H. Guo, and D. Jasnow, J. Stat. Phys. **41**, 353 (1985).
[17] Denjoe O'Connor and C. R. Stephens, Utrecht/DIAS Report No. THU-93/14, IJMPA (to be published).
[18] If a normalization condition with $L = \infty$ had been used, then temperature/field deviations would be measured relative to $T_c(\infty)$ or $\Gamma_c(\infty)$, the critical temperature of the bulk system or critical field of the $\beta = \infty$ quantum system, respectively.
[19] Denjoe O'Connor and C. R. Stephens, Nucl. Phys. **B360**, 297 (1991).
[20] M. Novotny, Phys. Rev. Lett. **70**, 109 (1993).
[21] F. Freire, D. O'Connor, and C. R. Stephens, J. Stat. Phys. **74**, 219 (1994).
[22] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).
[23] B. G. Nickel, J. Math. Phys. **19**, 542 (1978).