Group Velocity of Large Amplitude Electromagnetic Waves in a Plasma

C. D. Decker and W. B. Mori
Departments of Physics and Electrical Engineering, University of California, Los Angeles, California 90024
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The nonlinear group velocity of a short intense laser pulse propagating in a cold underdense unmagnetized plasma is examined. Analytical expressions for the group velocity are derived. These expressions reduce to the usual $\partial\omega/\partial k$ form for small amplitude and are verified for arbitrary amplitude using particle in cell simulations on a cyclic mesh. We find that the leading edge of a pulse moves at the linear group velocity and that the phase velocity of the excited wake is found to be less than the group velocity of the pulse. The techniques used can be applied to other waves in a plasma.

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It has been recognized since Rayleigh [1] that, besides the phase velocity v_ϕ, there are several velocities associated with a wave which have physical significance. Rayleigh defined the group velocity to be the velocity of the envelope of a beat pattern constructed from two waves (ω_1, k_1) and (ω_2, k_2). This velocity is given by $\frac{\omega_1-\omega_2}{k_1-k_2} = \frac{\Delta\omega}{\Delta k}$, which reduces to the often quoted result $v_g = \frac{\partial\omega}{\partial k}$ in the limit that $\Delta\omega$ and Δk approach zero. Other velocities associated with a wave are the energy transport velocity [1], signal velocity [1], and the packet velocity [1].

In particular, for a plane electromagnetic wave propagating in an unmagnetized plasma it is well known that the group velocity is given by $v_g = c\sqrt{1-\omega_p^2/\omega^2}$, and that the phase and group velocities are related by $v_g v_\phi = c^2$, where ω_p is the plasma frequency and ω is the wave's frequency. Furthermore, it can be shown that both the energy transport velocity and the packet velocity are equal to v_g. However, these relationships are only true in the limit of infinitesimal wave amplitude. To understand the complications which arise for finite wave amplitude, consider Rayleigh's definition of the group velocity in terms of the beat velocity of two weakly nonlinear waves. Each wave now satisfies a generic dispersion relation which depends on the amplitude of both waves $\omega_1(k_1, k_2, A_1, A_2)$ and $\omega_2(k_1, k_2, A_1, A_2)$. Therefore, calculating $\frac{\Delta\omega}{\Delta k}$ of two waves, or $\frac{\partial\omega}{\partial k}$ of one wave, is now ambiguous because it depends on whether A_1, A_2, or some combination of both is kept fixed. Furthermore, for a wave packet with a spectrum of frequencies, rather than a few discrete modes, it becomes impossible to even define a dispersion relation. As a result, there appears to be little in the literature concerning or even defining a nonlinear group velocity. An exception is the work of Lighthill [2] and Whitham [3] who considered systems with identifiable Lagrangians. They find that the only velocity with a well defined nonlinear counterpart is the energy transport velocity. In particular, Lighthill showed that the energy transport velocity is equal to $\partial\omega/\partial k$ when holding the Lagrangian density over ω fixed.

In this Letter we calculate for the first time the energy transport velocity of nonlinear electromagnetic waves propagating in unmagnetized plasmas. Besides being of fundamental importance to nonlinear plasma physics, it has practical implications for laser driven accelerator schemes. In either the beat wave (PBWA) [4] or laser wake field (LWFA) [5] schemes the large transverse electric fields of high intensity lasers are converted into longitudinal electric fields of plasma waves. The longitudinal waves must have a phase velocity very close to the speed of light c in order that accelerated particles and the wave do not dephase. The phase velocity of the plasma wave is generally assumed to be the group velocity of the light waves.

This Letter is outlined as follows. We first derive an expression for the energy transport velocity for long pulses using the conservation of energy equation. We then combine this equation with the quasistatic approximations to obtain a nonlinear group velocity for arbitrary pulse lengths. These results are then verified using particle in cell (PIC) simulations on a cyclic mesh. Last, the consequences of this work to the laser wake field scheme are discussed. We note that recently Kuehl *et al.* [6] made a weakly nonlinear analysis of the group velocity and wake excitation of short pulses. We present a fully nonlinear treatment. Furthermore, they concentrated on the times larger than the pump depletion time, while we examine the early time behavior which is more relevant to the LWFA.

To obtain an energy transport velocity we begin with Poynting's equation

$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) + \nabla\cdot\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B}) + \mathbf{J}\cdot\mathbf{E} = 0 \,. \quad (1)$$

Using the relativistic fluid momentum equation we can write

$$\mathbf{J}\cdot\mathbf{E} = \frac{\partial}{\partial t}\left(nmc^2\gamma\right) + \nabla\cdot\left(nmc^2\gamma\mathbf{v}\right) \,. \quad (2)$$

Substituting Eq. (2) into Poynting's equation yields the conservation of energy equation

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$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}+nmc^2(\gamma-1)\right)$$
$$+\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}+nmc^2(\gamma-1)\mathbf{v}\right)=0\,. \quad (3)$$

A local energy transport velocity can be found by noting the above expression is of the form

$$\frac{\partial}{\partial t}(U)+\nabla\cdot\mathbf{S}=0\,. \quad (4)$$

We average Eq. (4) over the high frequency oscillations ($\langle\ \rangle$) and define a local group velocity $\mathbf{v}_g\equiv\frac{\langle\mathbf{S}\rangle}{\langle U\rangle}$.

Furthermore, if we define the position of a finite pulse as the energy weighted expectation value $\overline{x}=\frac{\int dx\,xU}{\int dx\,U}$, where the integration is over the length of the pulse, then the velocity of the pulse, i.e., the group velocity, is given by $\overline{v_g}\equiv\frac{d}{dt}\overline{x}=\int dx\,xS/\int dx\,xU$ where the conservation of U is explicitly used.

Using these definitions, we find from Eq. (3) that

$$\mathbf{v}_g=\frac{\langle\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}+nmc^2(\gamma-1)\mathbf{v}\rangle}{\langle\frac{E^2+B^2}{8\pi}+nmc^2(\gamma-1)\rangle}\,. \quad (5)$$

To determine an analytic expression for the group velocity we consider one dimensional wavelike solutions of the form $f(x-v_\phi t)$. In the limit of electromagnetic pulses much longer than $2\pi c/\omega_p$, the longitudinal electric field can be neglected so that from Faraday's law $E=\frac{v_\phi}{c}B$. Using this relationship, we reduce Eq. (5) to

$$v_g=(2c^2/v_\phi)\,\frac{1+\frac{\omega_p^2}{\omega^2}\frac{\langle(\gamma-1)nv_x\rangle}{n_0}\frac{v_\phi}{\langle p_\perp^2\rangle}}{1+c^2/v_\phi^2+2\frac{\omega_p^2}{\omega^2}\frac{\langle(\gamma-1)\,n/n_0\rangle}{\langle p_\perp^2\rangle}}\,, \quad (6)$$

where $v_g=\mathbf{v}_g\cdot\mathbf{x}$, $p_\perp=\frac{eE_\perp}{m\omega}$.

Expressions for v_ϕ can be found in Akhiezer and Polovin [7] for both linearly and circularly polarized light. The expressions can be summarized as $v_\phi^2=\frac{1}{1-\omega_p^2/\omega^2\gamma_{\perp 0}}$, where $\gamma_{\perp 0}^2\equiv 1+\langle p_\perp^2/m^2c^2\rangle$. If we define a nonlinear parameter $p_0\equiv\frac{eE_0}{m\omega}$, where E_0 is the amplitude of E, then $\langle p_\perp^2/m^2c^2\rangle=p_0^2$ is used for circularly polarized light and $\langle p_\perp^2/c^2\rangle=p_0^2/2$ is used for linearly polarized light. The circular polarization expression is exact while the linear polarization expression is valid for $\frac{\omega_p^2}{\omega^2}\ll 1$ [7–9]. For circularly polarized waves $n=n_0$, so that Eq. (6) reduces to

$$v_g=\frac{c^2/v_\phi}{1+\frac{\omega_p^2}{2\omega^2}\frac{\gamma_{\perp 0}-1}{\gamma_{\perp 0}(\gamma_{\perp 0}+1)}}\,, \quad (7)$$

where $\gamma_{\perp 0}$ is defined above. For linearly polarized waves, this expression is valid to order $\frac{\omega_p^2}{\omega^2}$ and can be derived by Taylor expanding Eq. (6) . For small amplitude, i.e., $\gamma_\perp\sim 1$, we recover the usual relationships $v_\phi v_g=c^2$ and $v_g=\partial\omega/\partial k=c\sqrt{1-\omega_p^2/\omega^2}$.

However, for nonlinear amplitudes this simple relationship between v_ϕ and v_g does not hold and Eq. (7) is not recovered by simply differentiating the nonlinear dispersion relationship while holding γ fixed. The importance of the nonlinear corrections to the $v_\phi v_g$ relationship can be most easily demonstrated by examining $\gamma_g\equiv(1-v_g^2/c^2)^{-1/2}$. Assuming $v_g=c/v_\phi$ gives $\gamma_g=\sqrt{\gamma_{\perp 0}}\,\omega/\omega_p$ while Eq. (7) gives $\gamma_g\sim\sqrt{\frac{\gamma_{\perp 0}+1}{2}}\,\omega/\omega_p$ in the $\omega_p\ll\omega$ limit.

For pulses which are less than a few plasma wavelengths $2\pi c/\omega_p$ long, a plasma wake field is excited and the relationship $E=\frac{v_\phi}{c}B$ does not hold. However, if we denote the laser field with $E_\perp$ and the plasma wake field with $E_\parallel$, then $E_\perp=\frac{v_\phi}{c}B_\perp$ and $B_\parallel=0$. We are in the short pulse regime so we make use of the quasistatic approximation [8]. The quasistatic approximation consists of neglecting $\frac{\partial}{\partial\tau}$ in the continuity equation and in the longitudinal equation of motion after a mathematical transformation has been made from the (x,t) to the $(\xi=x-ct,\tau=t)$ coordinates. The result is a coupled set of nonlinear equations for the scalar potential Φ and the vector potential $\mathbf{A}$. We emphasize that using the conservation equation derived from the quasistatic equations by themselves, provides an incorrect value for the group velocity [10]. However, the quasistatic equations give correct amplitudes for the fields and v_ϕ so when these fields are substituted into Eq. (5) a correct value for v_g is obtained.

The quasistatic approximation gives $\frac{c^2}{v_\phi^2}=1-\frac{\omega_p^2}{\omega^2\chi}$, $nv_x=(n-n_0)c$, $\gamma=\frac{\gamma_\perp^2-\chi^2}{2\chi}$, and $E_\parallel=\frac{\partial\chi}{\partial\xi}$, where $\chi=1+\phi$ and $\gamma_\perp^2=1+a^2$, with $\phi\equiv e\Phi/mc$ and $\mathbf{a}\equiv e\mathbf{A}/mc^2$. We use these relationships and some algebra to obtain a local value for the group velocity of a light pulse in the $\frac{\omega_p^2}{\omega^2}\ll 1$ limit,

$$v_g=\frac{c^2/v_\phi}{1+\frac{\omega_p^2}{2\omega^2}\left\{\frac{1}{p_0^2}\left[\left(\frac{\partial\chi}{\partial\xi}\right)^2+\frac{\gamma_\perp^2+\chi^2-2\chi}{\chi}\right]-\frac{1}{\chi}\right\}}\,. \quad (8)$$

The functional dependence of χ is described by

$$\frac{\partial^2\chi}{\partial\xi^2}=\frac{1}{2}k_p^2\left(\frac{\gamma_\perp^2}{\chi^2}-1\right)\,, \quad (9)$$

where $k_p=\omega_p/c$. For short pulses χ is predominantly a plasma wave wake. This wake is the basis of the LWFA. The long pulse limit can be recovered by neglecting $\frac{\partial\chi}{\partial\xi}$ in Eqs. (8) and (9).

It has been shown [8] that ϕ grows on ω_p^{-1} time scales. In particular, $\phi\sim\left(p_0\frac{k_p\xi}{2}\right)^2$, where $\xi=0$ is the front of the pulse, so that $\phi\ll 1$ when $\xi<\frac{2c}{p_0\omega_p}$. Therefore, the local group velocity of the very front of a pulse, or the group velocity of ultrashort pulses $l_0\ll c/\omega_p$, can be obtained by Taylor expanding Eq. (8) in terms of

ϕ. Noting that $\frac{1}{p_0^2}(\frac{\partial \chi}{\partial \xi})^2 \sim \phi^2$ and neglecting terms of order ϕ^2 gives $v_g = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$. We therefore conclude that the group velocity for the leading edge of long pulses, as well as the group velocity for ultrashort pulses, is the linear group velocity irrespective of the wave amplitude. For intermediate length pulses, $l_0 \sim c/\omega_p$, it is difficult to obtain a closed form expression for the local v_g. In principle, χ must first be solved using Eq. (9) and then substituted into Eq. (8). The pulses group velocity is then obtained by averaging the local value over the entire pulse.

These analytical expressions were investigated using the electromagnetic particle-in-cell code ISIS which has recently been modified to include a cyclic mesh. The cyclic mesh is a technique for following short pulses by removing columns of cells from far behind the pulse and placing (cycling) them at the front of the pulse with fresh particles. The cycling rate is chosen so that the grid moves at a speed c. These simulations are done in the x-y plane with linear polarization in the z direction. We initialize a laser pulse in vacuum and let it propagate in the x direction into the plasma. The initial profile is of the form $\frac{eE_\perp}{m\omega_p c} = \frac{p_0}{c}\frac{\omega}{\omega_p}\sin^2(\frac{\pi}{2}\frac{x}{l_0})\sin(\frac{\omega_0}{c}x)$.

We first verified the expression for the phase velocity for both circularly and linearly polarized light. This was done by tracking the position of wave crests. The scaling with both γ and ω/ω_p was confirmed. The group velocity was measured by calculating the pulse's energy weighted expectation position defined by $\overline{x} = \frac{\int dx\, x E_\perp^2}{\int dx\, E_\perp^2}$ at every time step and then evaluating $v_g = \frac{d}{dt}\overline{x}$ at the end of the simulation. We weighted the position using $E_\perp^2$ rather than the entire energy U. It can be shown [10] that this gives an error to v_g on the order of ω_p^4/ω^4 in the long pulse limit. If this is not done in the short pulse limit, the calculated group velocity would have been artificially lower than the theoretical value because the wake left behind the pulse, being a space charge wave, has zero energy flux but a nonzero energy density. A typical simulation result is shown in Fig. 1(a) for $\omega/\omega_p = 5.0, p_0 = 3$, and $l_0 = 10c/\omega_p$. We see that in the vacuum region the curve is flat, indicating a group velocity very close to the speed of light. Thus, the numerical dispersion associated with the field solver is much less than the plasma dispersion. Inside the plasma the group velocity [the slope of Fig. 1(a)] remains very constant for many plasma periods, thus allowing for very accurate measurements. As the pulse propagates further in the plasma, pump depletion occurs. Pump depletion results from pulse distortion and a lowering of the pulse's frequency which leads to a reduction in v_g. However, from a simple energy conservation analysis it can be shown that the pump depletion time scales as ω^2/ω_p^2 so this does not affect the measurements done earlier on in time. In Fig. 1(b) we plot Eq. (7) along with the PIC results for a long pulse with a Gaussian rise and fall of $l_0 = 20c/\omega_p$ and a flat section

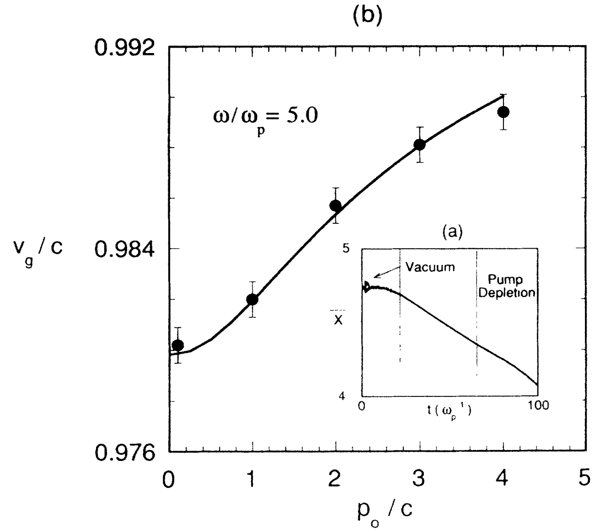

FIG. 1. (a) Weighted expectation position versus time from computer simulation. The slope of this curve is the group velocity. (b) Long pulse group velocity versus wave amplitude for $\omega/\omega_p = 5$. Data points are for simulations with pulse length of $140c/\omega_p$ and solid line is theory.

of $100c/\omega_p$. In the simulations the laser frequency was typically chosen to be $\omega/\omega_p = 5.0$ in order to lessen the computer time. Simulations were done for numerous values of p_0. From Fig. 1(b) we see excellent agreement between the theoretical expression, Eq. (7), and the PIC result. This agreement for the long pulse group velocity demonstrates that weighting the position with $E_\perp^2$ rather than U is accurate to order ω_p^2/ω^2. Simulations using higher values of ω/ω_p were also carried out, and the results were in agreement with theory.

Simulations of short pulses were also carried out to verify the implication of Eq. (8). The results are summarized in Fig. 2 where we plot the group velocity versus the pulse width for $p_0 = 3.0$ and $\omega/\omega_p = 5.0$. Based on Eq. (8) we expect the group velocity of a short pulse, i.e., $l_0 \leq c/\omega_p$, to approach the linear value. This is consistent with Fig. 2 where the dashed line through the leftmost points approaches the linear v_g. No simulation points are available for smaller values of l_0 because there are too few cycles within the pulse to define a single frequency.

The group velocity begins to increase as the pulse width increases. This occurs because v_ϕ decreases and the dominant term in Eq. (8) is the numerator. Physically, the increase in v_g is due to the reduction of the local value of ω_p caused by the density depression of the wake and the relativistic mass increase. The maximum value of v_g, therefore, occurs when the pulse resides entirely within the density depression. For longer pulse lengths parts of the pulse will again reside in density compressions. This leads to a reduction in the group velocity. This scenario is seen in Fig. 2 where the group velocity oscillates as a function of l_0. The periodicity corresponds

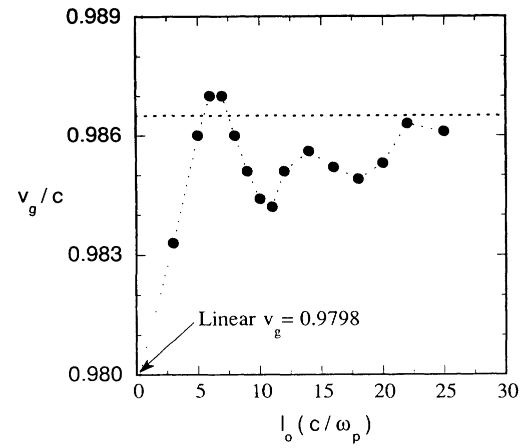

FIG. 2. Group velocity, v_g, versus pulse width for $p_0 = 3.0$ and $\omega/\omega_p = 5$. Dashed line is an $E_\perp^2$ weighted average of Eq. (7) over the pulse shape.

to the wake's wavelength which is a function of its amplitude and hence a function of p_0. Similar curves were obtained for other values of p_0.

As the pulse width increases further the group velocity asymptotes to the long pulse expression. The amplitude of the oscillation decreases because the wake's amplitude decreases with pulse length. The asymptotic limit is not that given in Eq. (7) because Gaussian shaped pulses were used in the simulations of Fig. 2. We therefore calculated an $E_\perp^2$ weighted average of Eq. (7) over the pulse shape and this value is plotted as the horizontal dashed line. The agreement between the calculated value and the rightmost simulation points is excellent.

A crucial issue for the laser wake field accelerator is the dephasing between the particles and the wake. To avoid dephasing, v_w should be as close to c as possible. Previously, it has always been assumed that $v_w = v_g$. However, this relationship can be altered by pulse shaping, linear (nonlinear) dispersion, photon acceleration (deceleration), and pulse distortion. We have carried out simulations to investigate the nonlinear dependence between v_g and v_w. The wake's phase velocity was determined by tracking the first minimum of $E_\parallel$. Sample results are presented in Fig. 3 where v_w and v_g are plotted versus p_0 for $\omega/\omega_p = 5.0$ and $l_0 = 6c/\omega_p$.

We find that $v_w = v_g$ only for linear values of p_0 and symmetric pulses. However, as shown in Fig. 3, as p_0 increases v_g increases while v_w decreases. This opposite dependence on p_0 is not paradoxical because the part of the pulse which is generating the wake need not travel at the average velocity of the pulse. We hypothesize that the point at the front of the pulse which generates the wake gradually etches backward due to local pump depletion.

As a result, wakes excited by the leading edge of the laser should cause the excitation point to etch backward while wakes excited by the trailing edge of the laser should cause the excitation point to etch forward. We expect, therefore, $v_w > v_g$ for pulses with a long rise time and sharp fall and $v_w < v_g$ for pulses with a sharp

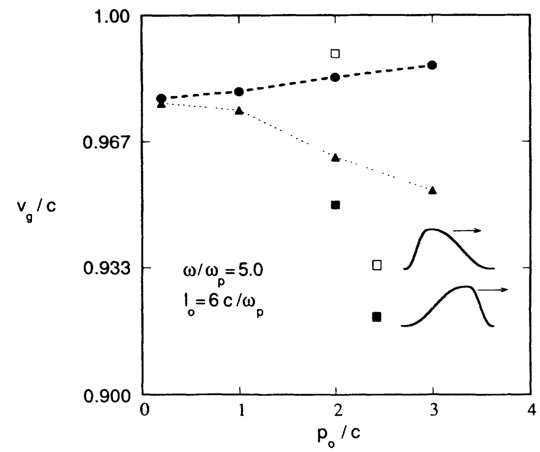

FIG. 3. Wake field phase velocity v_w (triangles) and pulse group velocity v_g (circles) versus amplitude p_0 for symmetrically shaped pulses and v_w (squares) for asymmetrically shaped pulses.

rise and a long fall. Indeed this is what is observed in simulations of such pulses as shown in Fig. 3.

To illustrate the importance of this decrease in v_w for symmetric pulses in possible near term experiments, we simulated $\omega/\omega_p = 20$ and $p_0 = 2.0$. This corresponds to a 35 fs/1 μm laser pulse with $I = 5 \times 10^{18}$ W/cm^2 propagating through a plasma of $n = 4 \times 10^{18}$ cm^{-3}. We find that $\gamma_w^2 = 230$ while the analytic $\gamma_g^2 = 550$. Therefore, the maximum energy gain is half of what is naively expected.

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[10] C.D. Decker and W.B. Mori (to be published); for example, the quantum static equations give $\gamma_g = \sqrt{2\gamma_{\perp 0}^2/(1+\gamma_{\perp 0})}\ \omega/\omega_p$.