Route to Vortex Reconnection

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The behavior of two antiparallel vortices of equal strength, moving in an ideal fluid, is studied numerically. The system is unstable and after a transient period two points of the vortices collide. On their way to the collision the two vortices form a pyramidal structure which is independent of the initial conditions. The connection process tends to follow a universal route for all kinds of initial vortex-antivortex arrangements.

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The interaction of vortices plays a special role in the study of ideal fluid hydrodynamics. Building on the pioneering work by Schwarz [1] and Siggia [2], we investigated numerically the motion of two vortex contours of equal strength moving in their induced velocity fields assuming ideal fluid properties. The simulations concentrate on the evolution in the regions of opposite vorticity. This is of interest for the understanding of the motion of vortex rings in general and of vortices in the vicinity of a liquid or solid interface where the motion is determined by the interaction with the mirror vortex. Furthermore vortex-antivortex interactions play a key role in vortex tangles such as in superfluid ^{4}He where the reconnection process forms, together with the local term, one of the key ingredients of the powerful reconnecting vortex-tangle model [3]. Finally, we discuss the implications of our results for the so-called conservation of helicity [4] and the possible existence of singular solutions of the Euler equations.

The results of the calculations described in this Letter are of general validity, but we will treat the motion of quantized ^{4}He vortices as an example because superfluid ^{4}He (with vortex strength $\kappa=1.0\times10^{-7}$ m^2/s) almost satisfies the conditions for an ideal fluid and the vortices are always of equal strength. We studied vortex contours with a typical radius of 10 μm, and stopped the calculations whenever the distance was equal to the effective core diameter δ of 0.1 nm. According to Ting and Klein a deviation of about 2% of the asymptotic value is to be expected when the two vortex lines are within a distance 3δ of each other [5].

The dynamical behavior was studied numerically by integrating the Biot-Savart solution for the ideal-fluid velocity field

$$\mathbf{v}(\mathbf{r})=\frac{\kappa}{4\pi}\int\frac{(\mathbf{s}-\mathbf{r})\times d\mathbf{s}}{|\mathbf{s}-\mathbf{r}|^3}, \quad (1)$$

where κ is the constant strength of the vortex. The integral is taken along the vortex lines. This is a well-defined quantity when the vorticity is confined to a very thin tubular region in space.

If Eq. (1) is evaluated for a point $\mathbf{r}$ on the vortex line it diverges logarithmically. Following Moore and Saffman [6] and Schwarz [1] a region of constant length δ, adjacent to $\mathbf{r}$, is excluded from the integration. The magnitude of δ is on the order of the diameter of the core radius a_0 and chosen in such a way that the velocity of a vortex ring is in agreement with experiment. The remaining part of the integral is divided in a region near $\mathbf{r}$ with a length which is typically 20% of the local radius of curvature. This (local) contribution to the velocity can be calculated analytically. The rest of the vortex line is integrated numerically. The discretization point distance is typically 20% of the local radius of curvature or $\frac{2}{3}$ times the distance of the nearest point on the other vortex (whichever is smaller).

Figure 1(a) represents the projection in the XY plane of two vortex contours which initially were identical circles of 10 μm radius in the XY plane at a distance of 2 μm [Fig. 2(a), case 1]. As a result of the interaction between the two contours, as described by Schwarz [1], and in more detail by Liu, Tavantsis, and Ting [7], the vortices have developed kinks in the points where they are close together. The figure is symmetric. This is obvious since the configuration was symmetric at the start. On the scale of Fig. 1(a) the vortices seem to be connected in the center forming the letter "X." Therefore this point is called the X point [8]. In three dimensions the vortices form the top of a pyramid with a rectangular base plane with a typical base-line ratio of 4 to 1. Figure 1(b) is a blowup of the region near the X point. This structure has been found before by other workers [1,2,8–10]. On this scale the curves form two hyperboles, each with two asymptotic lines.

The evolution of vortex contours was calculated for several different initial conditions [Fig. 2(a)]. After a transient period a structure develops which is the same as the one described in Fig. 1(b). The pyramid is *symmetric* even for asymmetric starting conditions [Fig. 2(b)]. The angles between the asymptotic lines forming the pyramid are given in Fig. 2(c) as functions of the ratio between the minimum distance D of the two vortices [see Fig. 1(b)] and δ. The radius of curvature at the top is $D\arctan^2(\frac{1}{2}\phi_1)/\sin(\frac{1}{2}\phi_2)$. The values of the angles are determined in regions where the curves are practically straight, in fact, where the radius of curvature is 2 orders

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of magnitude larger than at the top. After the transient period (where the angles are not well defined) the angles turn out to follow the same D/δ dependence. This means that the geometry, presented in Fig. 1(b), is independent of the starting conditions. This behavior is characteristic for the self-similar behavior as described by Barenblatt and Zel'dovich [11]. Only the orientation and the position in space of the structure are determined by the initial conditions.

The dependence of the angles on D/δ can be fitted by a linear $\ln[\ln(cD/\delta)]$ relationship with $c \approx 1.3$. The intervortex angle ϕ_1 changes from about 135° to 115° and the intravortex angle ϕ_2 only with a few degrees on a 25° level, while D/δ changes about 2 orders of magnitude. If the angles would be constant the time evolution could be described simply as a rescaling of Fig. 1(b) which would correspond with perfect self-similarity. The geometrical evolution is independent of the vortex strength κ, which only determines the time scale of the process. The characteristic time is δ^2/κ; the characteristic length scale is the cutoff length δ.

An equal geometry implies an equal evolution with time t. This was confirmed by the calculations (not shown here). Near the kinks the velocities are much larger than in the other parts of the vortex contours so the short-term evolution is determined by the behavior near the kinks. Figure 3 shows D and D^2 as functions of time for case 1 of Fig. 2(a). The velocities near the X

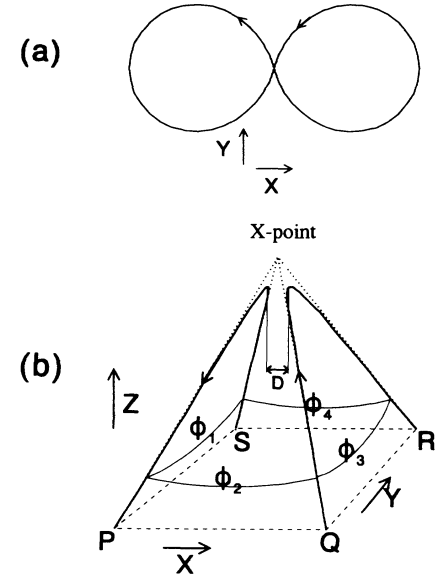

FIG. 1. XY projection of two vortices, originally in the XY plane [case 1 of Fig. 2(a)], on their way to reconnection. (a) Projection on the XY plane. On this large scale the two rings seem to be connected in the X point. (b) Same situation as in (a), but magnified, and looking from a different angle. The two approaching vortices form two hyperbolic curves; the asymptotic lines form a pyramid. The angles ϕ_1 and ϕ_3 are the intravortex angles PXS and QXR; the angles ϕ_2 and ϕ_4 are the intervortex angles PXQ and SXR. The minimum distance is D.

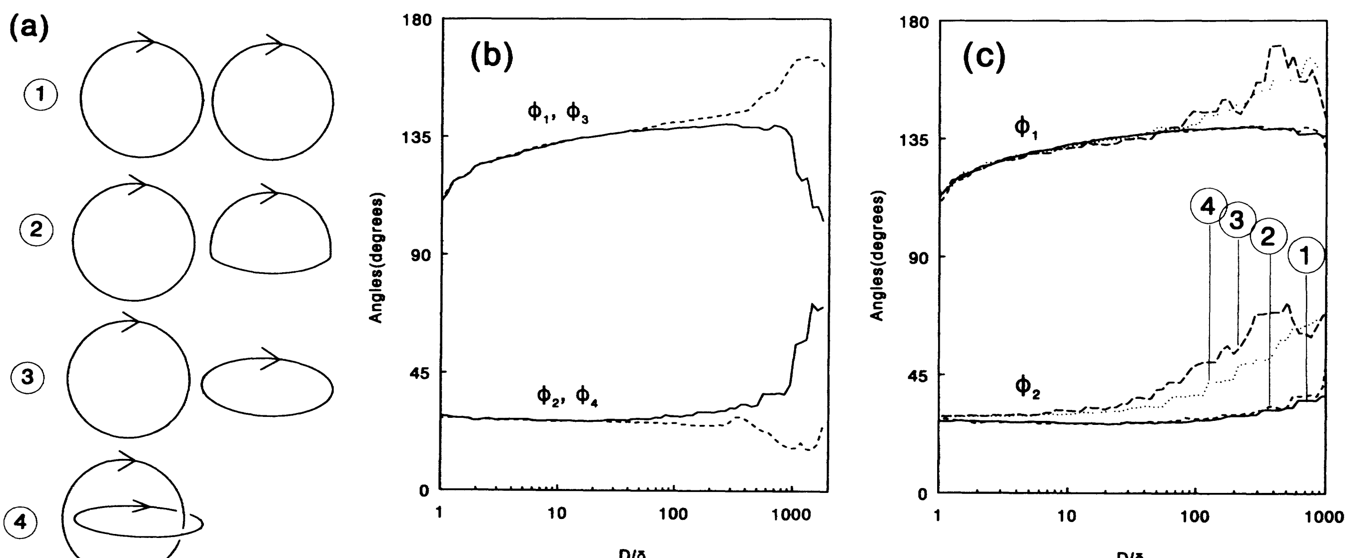

FIG. 2. Top angles of the pyramid as functions of D/δ for various initial conditions. The arrows in the curves indicate the direction of the vorticity. (a) The starting contours: 1: two circles with equal diameters in one plane; 2: one circle and one contour made up of a half circle and a half ellipse; 3: two circles of equal diameters, the planes making an angle of 60°; 4: two linked circles. (b) D/δ dependencies of the angles [defined in Fig. 1(b)] between the asymptotes in the (asymmetrical) case 2 of (a). The angle ϕ_1 tends to be equal to ϕ_3, and ϕ_2 tends to ϕ_4. This leads to a symmetrical pyramidal structure. (c) D/δ dependencies of ϕ_1 and ϕ_2. The circled numbers refer to the four cases of (a) (——: case 1; ---: case 2; — — —: case 3; ····: case 4). After a transient period the angles tend to the same $\phi - D/\delta$ dependence.

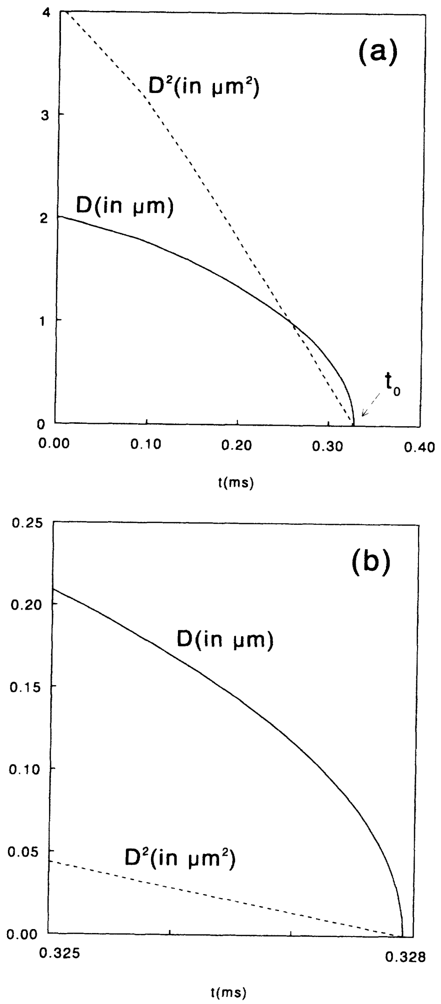

FIG. 3. Time dependence of D and D^2 for case 1 of Fig. 2(a). (a) The large time scale; (b) the small time scale.

point *increase* when the distance between the two vortices decreases. The distance tends to zero in a well defined moment t_0 in time and at a well defined point in space: The two vortices collide. In agreement with the results of Siggia [2] the D^2-t dependence is practically linear even just before the collision [Fig. 3(b)].

The linear D^2-t dependence can be understood assuming that the evolution and the geometry of the vortex lines are given by

$$\mathbf{s}(t,\xi) \approx \mathbf{s}_X + f(t)\boldsymbol{\sigma}(\xi) , \qquad (2)$$

where $\mathbf{s}_X$ represents the position of X point, f is a scaling factor, and $\sigma(\xi)$ a curve parametrized by the parameter ξ. The main time dependence is determined by the scaling factor f; both $\mathbf{s}_X$ and $\sigma(\xi)$ are weakly time dependent. The picture behind Eq. (2) is that the tops of the vortex curves move towards the X point (f decreases with time). From Eq. (2) and Eq. (1) it follows that $f^2 \sim t - t_0$ [12], so $D^2 \sim t - t_0$. From the slope of about $\kappa/2\pi$, determined from Fig. 3(b), one obtains $D^2 \approx (\kappa/2\pi)(t-t_0)$. In general the velocities just before the collision will be very high. For $D=\delta$ the velocity $dD/dt \approx \kappa/4\pi\delta$. In the case of quantized ^{4}He vortices this amounts to 80 m/s. However, at velocities on the order of the speed of sound the assumptions underlying Eq. (1) are no longer satisfied.

So far the description basically holds for any two line vortices of equal strength in ideal fluids. Also in this respect the route to the collision is universal. However, if $D \approx \delta \approx a_0$, one has to take the detailed properties of the vortex core into account such as the vorticity distribution, viscous effects, density variations (vacuum core?), or even quantum effects (^{4}He). In fact the structure of the core itself will be affected [8-10]. Although these processes are hard to generalize, it seems that in any case the topology of the system of vortex contours has to change since otherwise the evolution cannot continue beyond the moment of the collision. In the case of ^{4}He it is assumed that the process is nondissipative [13]. The collision is assumed to be followed by a cross linking of the vortex elements. A similar disconnection occurs in other media [8-10]. The fact that the geometry of the vortex structure before the collision is universal implies that the evolution immediately after the collision will be universal too.

Case 4 in Fig. 2(c) represents two linked vortex rings. For ^{4}He, assuming that initially the diameters of the rings are 20 μm and that they are 2 μm apart, the vortices will collide after 0.17 ms. Such a collision, followed by a cross linking in the way described above, would imply that this system of vortex contours is subject to a topological change. In general one may raise the question of what the law of *invariance of helicity* [4] contributes to our understanding of turbulent phenomena: When the two vortex cores do not touch the conservation is trivial by definition, when they touch it is a meaningless quantity, and when the process is all over the helicity has changed.

The collision process described in this Letter is related to the (near) singular behavior of the solutions of the Euler equations [14]. In this Letter we assumed a finite effective core size δ. The behavior would be really singular if δ would tend to zero while κ remains finite. Physically, this is impossible because the vortex would have infinite energy, infinite self-velocity, and an infinite negative pressure in the center of the vortex. Mathematically, one can consider the case where δ is infinitesimally small. In that case the collision of the vortex lines corresponds with an intersection of two vortex lines, which can then reconnect and continue the evolution. However, δ is the scaling parameter of the phenomena, so it may be questionable to take the limit of δ to zero, even mathematically. The most interesting and realistic cases are situations where κ and δ both tend to zero in a well-defined way (e.g., with κ/δ^2 constant). This is the case when a finite core is represented by a collection of infinitesimal vortex filaments. It is conceivable that collisions and subsequent reconnections can take place between individual filaments. It would be most interesting to investigate whether this can be a mechanism for reconnection of classical macroscopic vortices without viscous dissipation.

In summary we have shown that the system of two antiparallel vortices of equal strength is unstable. The interaction is a universal and elegant process that tends to follow the same and well-defined route for many different initial conditions. It ends in a collision between the two vortices.

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