Periodic, Quasiperiodic, and Chaotic Localized Solutions of the Quintic Complex Ginzburg-Landau Equation

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We discuss time-dependent spatially localized solutions of the quintic complex Ginzburg-Landau equation applicable near a weakly inverted bifurcation to traveling waves. We find that there are—in addition to the stationary pulses reported previously—stable localized solutions that are periodic, quasiperiodic, or even chaotic in time. An intuitive picture for the stability of thee time-dependent localized solutions is presented and the novelty of these phenomena in comparison to localized solutions arising for exactly integrable systems is emphasized.

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Ever since the discovery of solitons [1] there has been considerable interest in localized solutions for spatially extended systems. In addition to the solitons of dispersive integrable systems [2,3], stable localized solutions have been found to occur in a system with both *dissipation* and *dispersion*, namely, the complex Ginzburg-Landau equation with a destabilizing cubic term and a stabilizing quintic term [4–7], which is a generic equation describing systems near a weakly inverted (subcritical) bifurcation to traveling waves. These localized solutions have a fixed shape for the modulus. They can be considered as the analogs of the localized solitons of the nonlinear Schrödinger equation [2,3], where the shape for the modulus is also fixed. In contrast to solitons in completely integrable systems, the pulses found for the coupled quintic complex Ginzburg-Landau (CGL) equations cannot only interpenetrate each other with their size and shape unchanged after the collision, but they can also annihilate each other, depending on the values of the cross coupling for counterpropagating waves [5,6]. Experimentally, stable localized solutions in one spatial dimension with an envelope that has fixed shape have been observed in binary fluid convection [8–10], which is a system exhibiting an inverted bifurcation.

When going from one to two spatial dimensions the situation changes drastically. While for most integrable equations two-dimensional localized solutions are unstable, the quintic CGL equation appropriate for an anisotropic two-dimensional system allows for the existence [4,11] of stable localized two-dimensional solutions. Their interactions, which have also been studied recently [11], are found to show even more different possibilities, since one has an additional parameter compared to the one-dimensional case: the impact parameter, i.e., the vertical distance between two localized solutions well before the collision.

While no systematic experiments have been performed to test the predictions for anisotropic systems, recent detailed experiments on binary fluid convection in a circular container [12] reveal the existence of two-dimensional (2D) localized objects as long transients. Whether it is possible to get stable spatially localized 2D pulses in this system is unknown.

It is also important to note that the stability of 1D and 2D localized solutions is not linked exclusively to nonlinearities of the polynomial type in the envelope equations. Very recently we have shown [13], that the amplitude equation applicable to a dye laser with saturable absorber in the good cavity limit also allows for the existence of stable 1D and 2D solutions, although the nonlinearity in the corresponding evolution equation is of the saturation type.

Other fluid systems which exhibit subcritical behavior are pipe and channel flow. These systems are found to exhibit three-dimensional turbulent localized structures which slowly spread with time—the turbulent slugs first observed by Reynolds in 1883 in pipe flow experiments [14] and turbulent spots observed in plane channel flow [15]. In addition to the turbulent slugs which slowly spread with time, turbulent localized structures which keep a fixed shape on average and do not decay or spread with time have also been observed in pipe flow [16]. These structures are referred to as turbulent puffs.

It has also been known for some time [17,18] that chaotic localized solutions, which slowly spread with time, exist for the quintic complex Ginzburg-Landau equation. So an interesting question to ask is whether or not chaotic localized solutions which keep a fixed shape on average (i.e., do not spread or decay with time) exist as solutions of this equation.

In this paper we show for the first time that localized solutions, which breathe in the modulus and, in contrast to the localized solutions reported previously [4–7], have *no* analog in integrable systems, exist for the quintic CGL equation. This breathing motion can be periodic, quasiperiodic, or chaotic depending on the parameter values.

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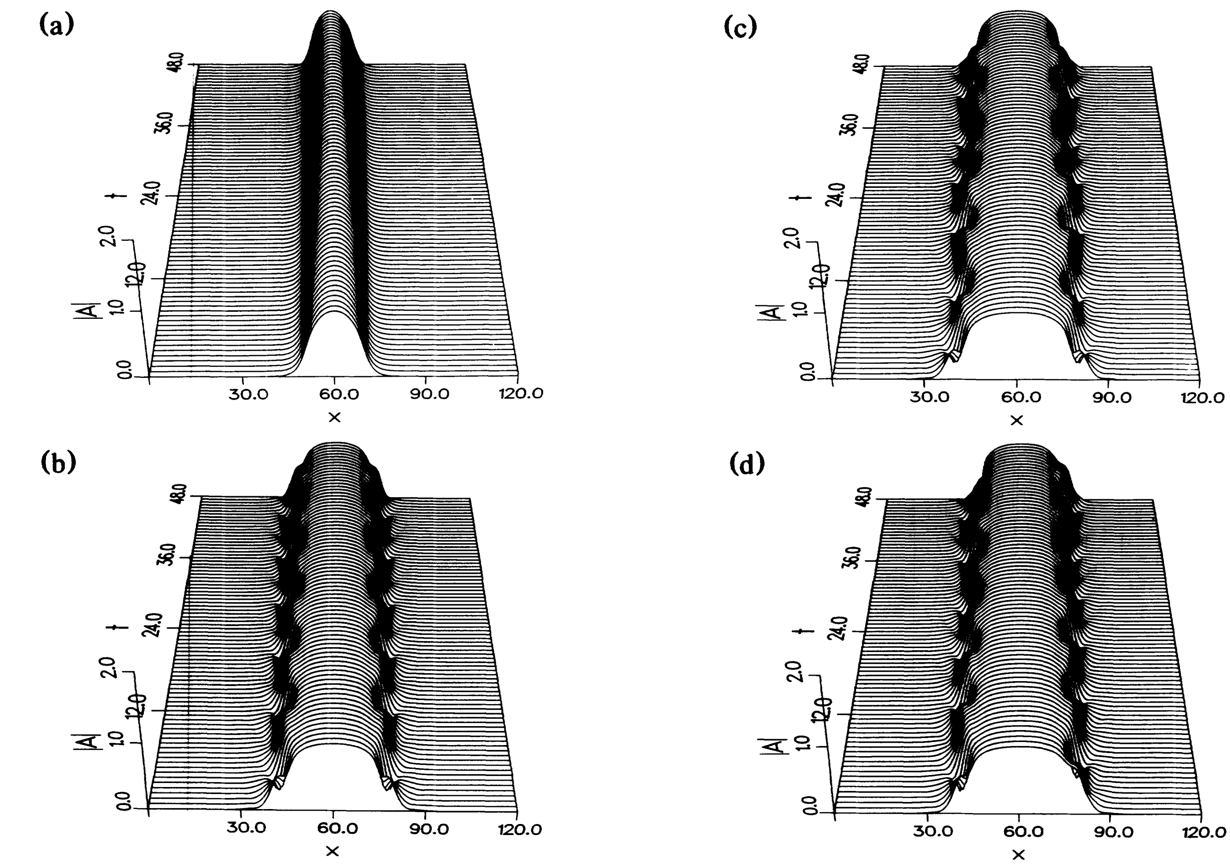

FIG. 1. Three-dimensional space-time (x-t) plots for the magnitude of the amplitudes for the different types of stable spatially localized states we have observed. The parameter values are $\chi=-0.1$, $v=0$, $\gamma_i=-1.1$, $\beta_r=-3.0$, $\beta_i=-1.0$, $\delta_r=2.75$, $\delta_i=-1.0$; they are identical to those used in previous publications on the subject (Refs. [3-5]) except for the diffusion coefficient γ_r and the fixed value of γ_i. (a) $\gamma_r=1.200$ (stationary localized state). (b) $\gamma_r=0.900$ (periodic localized state). (c) $\gamma_r=0.876$ (period-24 localized state). (Note that only about 6 periods are shown in the plot.) (d) $\gamma_r=0.873$ (chaotic localized state).

We are also not aware of any experimental observations of stable localized states of these types. These solutions are very different from the "breathers" of the integrable sine-Gordon equation, which simply oscillate periodically about zero for the real field and which, in fact, are more closely related to the solitons of the nonlinear Schrödinger equation, which oscillate periodically about zero for the real and imaginary parts, although the modulus is fixed. The solutions we observe in the quintic CGL equation oscillate for the modulus about some fixed shape. We note that for reaction-diffusion models for two real variables, spatially localized solutions that breathe periodically have been reported [19,20]. However, these localized solutions, which arise as the consequence of a delicate balance between the diffusivities and the ratio of reaction rates, are very different in character and result from a very different mechanism than the localized states studied in this paper.

The equation we study is

$$A_t+vA_x=\chi A+\gamma A_{xx}-\beta|A|^2A-\delta|A|^4A\,, \quad (1)$$

where A is a slowly varying complex amplitude and the coefficients (except for the group velocity v) are in general complex, i.e., of the form $z=z_r+iz_i$. The coefficient χ may be taken as real, since the imaginary part can be transformed away with a simple transformation. Also v may be taken as zero by transforming into a moving frame of references. We note, however, that this transformation is no longer straightforward when the influence of noise is considered [21]. We take $\chi<0$ and $\beta_r<0$ so that the system is subcritical and take $\delta_r>0$ to guarantee saturation. In writing down Eq. (1) we have discarded nonlinear gradient terms [6,22,23].

Figure 1(a) shows a space-time plot of the modulus for parameter values for which the modulus is stationary. This state was created by perturbing the state $A=0$ with a Gaussian of sufficient amplitude. If the amplitude is too small the solution damps to zero (since the system is subcritical). In the asymptotic time limit the system settles to the state shown. We note that, even though the modulus is stationary, the real and imaginary parts of A oscillate periodically about $A=0$. There are two factors which are responsible for the stability of this solution. One is that there are two basins of attraction for spatially

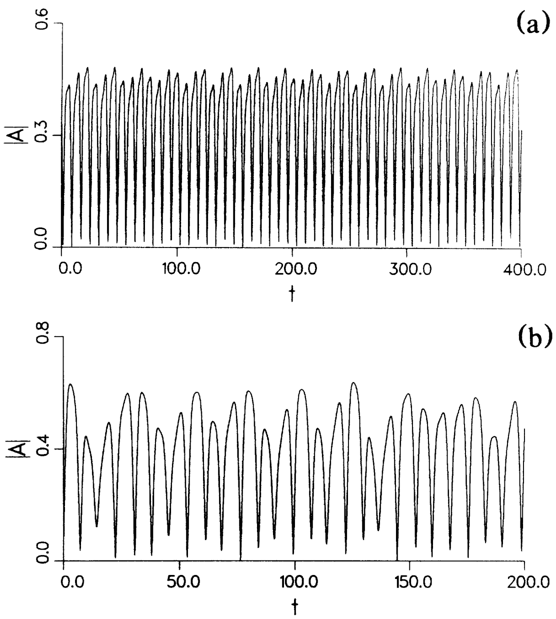

FIG. 2. The time series in the asymptotic regime is shown for the modulus in the quasiperiodic and the chaotic states. The parameters are as in Fig. 1. The time series was taken at $x=79.8$. (a) Quasiperiodic localized state: $\gamma_r=0.880$. (b) Chaotic localized state: $\gamma_r=0.873$.

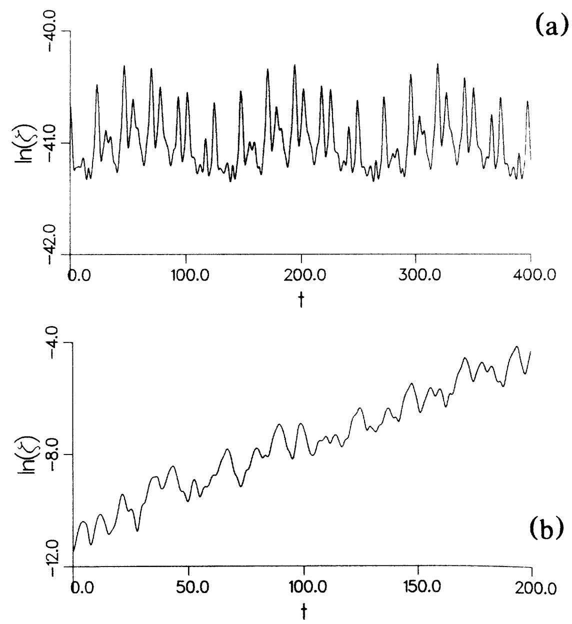

FIG. 3. The separation ζ in the asymptotic regime (after initial transients have died out) is shown for the quasiperiodic and the chaotic states. The parameters are as in Fig. 2. (a) Quasiperiodic localized state: $\gamma_r=0.880$. (b) Chaotic localized state: $\gamma_r=0.873$.

uniform solutions (assuming $\chi<0$, $\beta_r<0$, and $\delta_r>0$). Therefore it is possible to have parts of the solution lying in one basin of attraction and the surrounding parts of the solution lying in the basin of attraction $A=0$, with an interface between them [18]. The second is due to a nonvariational effect [4]. If the coefficients of the equation are real, the system is purely dissipative, a Liapunov functional exists, and there are two potential wells, assuming $\chi<0$, $\beta<0$, and $\delta>0$. In this case, at best there can be a neutrally stable solution with part of the system lying in one potential well surrounded by the rest of the solution lying in the potential well at zero. Only if there is in addition dispersion can stable localized solutions exist for a parameter regime of nonzero measure.

We now gradually decrease γ_r (i.e., decrease the dissipation) compared to its value in Fig. 1(a) ($\gamma_r=1.20$) keeping the other parameter values fixed. As γ_r is decreased, the solution becomes wider. At some point ($\gamma_r=1.091$) the stationary localized solution becomes unstable, and the system makes a continuous transition to a time-periodic state. (By continuous we mean that no hysteresis was observed at the transition when changing γ_r by 10^{-3}.) Figure 1(b) shows a space-time plot of the modulus for a stable periodic state ($\gamma_r=0.9$). By stable we mean that the solution lies on the periodic attractor. As can be seen from the figure, the modulus breathes in a periodic fashion with time. More specifically, the breathing motion can be described as follows. As time progresses, the solution increases in width. At some time indentations occur symmetrically on both sides of the solution. These indentations grow and travel outward, forming wings which then decrease in amplitude and eventually damp. The reason the wings damp is that they are sufficiently small in amplitude and that the system is subcritical.

For values of γ_r between $\gamma_r=0.885$ and $\gamma_r=0.880$, we find that the asymptotic state can depend on the initial conditions (e.g., Gaussian or final state of previous run) as well as on the magnitude and sign of the step size made in γ_r. Among the states observed are period-1, -3, -6, -12, and -24 states as well as quasiperiodic states. In Fig. 1(c) we give as an example a space-time plot of a stable period-24 localized solution. It is seen that the behavior is very similar to that of the periodic state—with breathing and damping of side wings—except that the motion is period-24 in time. Figure 2(a) shows a plot of a time series of a quasiperiodic state at a fixed spatial point for the asymptotic state for $\gamma_r=0.880$. It is seen that the motion is not periodic. In order to demonstrate that the motion is not chaotic in time, the separation between two nearby trajectories is calculated [17]. Figure 3(a) shows a plot of the logarithm of the separation. Since the average separation is constant (i.e., the largest Liapunov exponent is zero), the system is not chaotic. Since the motion is neither stationary nor periodic and also not chaotic, it is quasiperiodic. Again by stable we mean that this solution lies on the quasiperiodic attractor.

As γ_r is further decreased the system makes a transition to a stable localized chaotic state ($\gamma_r=0.8745$). By stable we mean that the solution lies on the chaotic attractor (we performed our runs for 300000 iterations with a time step of 0.01). We note that no hysteresis was observed at the chaotic transition when changing γ_r by 5×10^{-4}. Figure 2(b) shows a time series for $\gamma_r=0.873$ at a fixed spatial point. It is seen that the time series is indeed irregular. In order to determine whether the motion is chaotic, the logarithm of the separation between nearby trajectories is plotted as a function of time [see Fig. 3(b)]. It is seen that, since the trajectories separate exponentially on the average in time (i.e., the largest Liapunov exponent is positive), the motion is indeed chaotic. Figure 1(d) shows a space-time plot of this stable chaotic localized solution for large times. Again it is seen that there are some similarities with the periodic states—breathing and damping of side wings—except that, in addition to the motion being chaotic in time, the solution is no longer spatially symmetric. Initially the chaotic solution was oscillating symmetrically, but because the state is chaotic (i.e., nearby states separate exponentially on the average), the slight asymmetry due to roundoff error was exponentially amplified causing the solution to eventually become asymmetric as seen in the figure. This symmetry breaking process can be accelerated by adding noise to the system.

As long as the side wings are sufficiently small, they will damp and the solution will remain localized in time. As γ_r is further decreased the solution no longer remains localized, but fills in. This occurs at $\gamma_r=0.871$. The reason the solution spreads and fills in for these parameter values is that the amplitude of the side wings that form are sometimes large enough to grow instead of damp. Therefore these side wings will grow and cause the solution to spread and fill in. This spreading mechanism is similar to what occurs in the spreading chaotic localized solutions studied previously [17,18].

As is well known, a route to chaos from stationary via periodic and quasiperiodic states has been previously discussed for systems without spatial degrees of freedom [24]. Here we have shown for the first time that such a route to chaos can occur in a spatially inhomogeneous pattern and, in particular, for a pattern which is spatially localized.

In conclusion, we have found that stable spatially localized periodic, quasiperiodic, and chaotic solutions exist for the quintic complex Ginzburg-Landau equation. In contrast to the stationary solutions for this equation reported previously [4–6], which have an analog in the solitons of integrable systems, the time-dependent solutions which we studied here have *no* analog in integrable systems. Likely candidates to find such solutions experimentally appear to be binary fluid convection and electroconvection in nematic liquid crystals, both of which exhibit a subcritical bifurcation to traveling waves [8,9,25]. It will be very interesting to see whether such localized solutions can be found in experiments.

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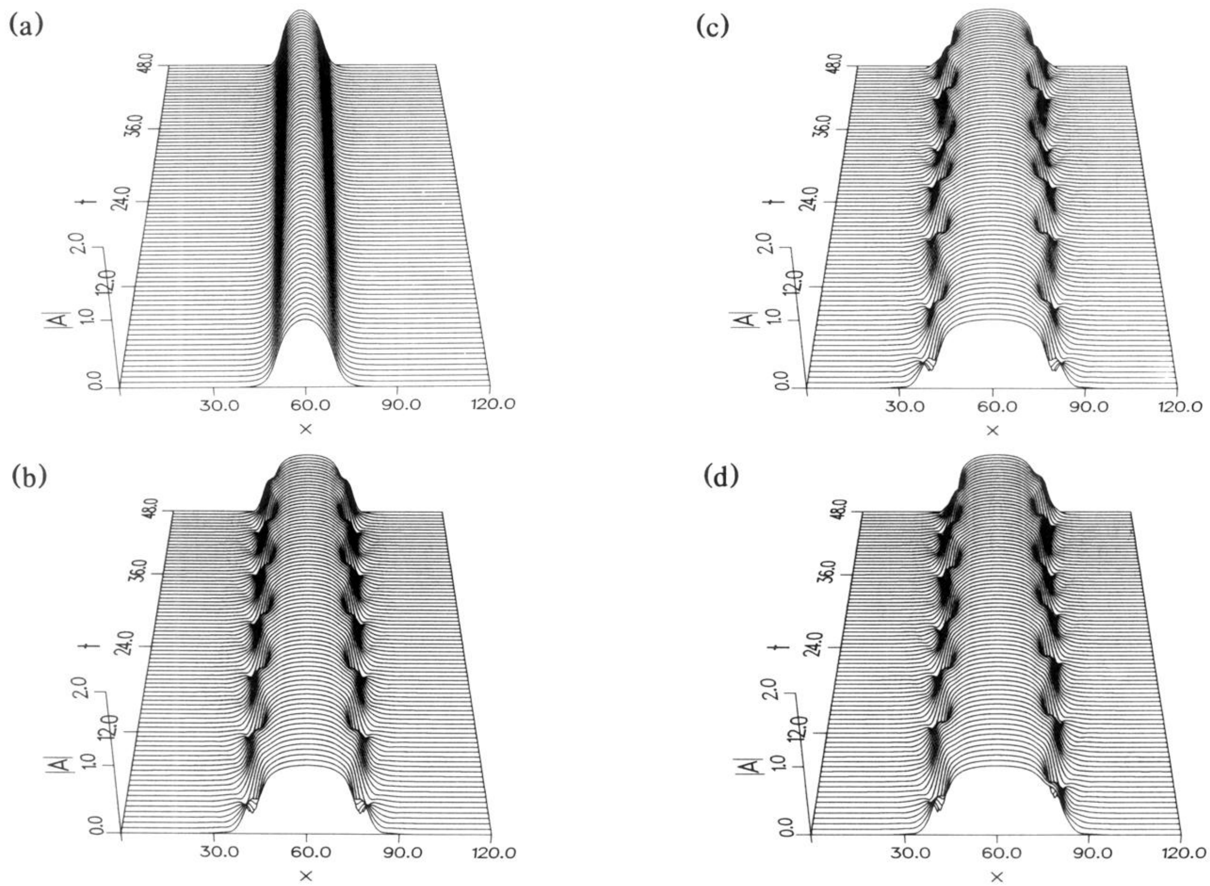

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