Stochastic Fluctuations and Structure Formation in the Universe

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It is shown that the evolution of the density perturbations during certain eras of substantial entropy generation in the Universe can be described using the Kardar-Parisi-Zhang equation. Therefore, the influence on cosmological structure formation by stochastic forces arising from various dissipations can be studied through the universal characteristics of surface growth in 3 + 1 dimensions. We identify eras of strong stochastic fluctuations and describe dynamically how these other dissipative sources of noise, besides initial (inflationary) quantum fluctuations, generate seeds of density perturbation with power law spectrum, including the Harrison-Zeldovich spectrum.

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In the last few years, significant progress has been made in understanding the dynamics of growing a rough or structural surface from an initially flat surface by random fluctuation [1]. Many structure formations in physics have been understood by studying the scaling properties of their growth patterns. Central to these studies has been the characterization of universal properties associated with systems of diverse physical attributes. To guide in these investigations, a major breakthrough has been the development of systematic analytic treatments inspired by scaling and renormalization group theory [2]. This treatment, aimed at studying the spatial and temporal behavior of structural growth, has revealed that the universal scaling properties come from the nonlinear and stochastic terms in the dynamical equation.

In principle, the structure formation in the Universe can also be classified as the phenomena of structural "surface" growth. Big-bang cosmology essentially tries to explain how an initially homogeneous mass distribution evolved into its present inhomogeneous state. In the language of the spacetime metric, it explains how an initially flat or smooth three-dimensional surface described by the Robertson-Walker metric evolved into a wrinkled one. The analogy to surface formation takes root by associating the initial mass distribution with a flat three-dimensional surface and its subsequent structure formation as that of surface roughening. This analogy gains interest by noting that cosmological structure also shows scaling in, for example, two-point correlation functions of galaxies, clusters of galaxies, and quasars, all of which behave as $r^{-\gamma}$ with $\gamma \sim 1.8$ up to present day scales of about 300 Mpc.

In the standard inflationary model, it is assumed that the seeds of the density perturbation are produced by the quantum noise of scalar fields during the inflation era [3]. In this model scaling structure is therefore explained as due to white noise seeds from this quantum fluctuation. However, besides quantum fluctuations, there are also time periods when stochastic fluctuations are large and which can, as we will see, lead to scaling seeds. Although this connection of stochastic fluctuations to scaling makes it of special interest, it should be recalled that quite generally dissipations must be accompanied by fluctuations or stochastic forces. In cosmology much work has concentrated on the effects of dissipation, for instance during the reheating period. This dissipation by the damping of scalar fields must also imply fluctuations of it, which preempts the investigation in this paper.

Analytic studies have shown that scaling behavior is common to systems that obey nonlinear dynamical equations with also a stochastic driving term [1]. The structure formation in the Universe, in particular at subhorizon scales, is just such a system. Therefore, it is worthwhile to study the models of cosmological structure formation from the point view of the universal dynamics that governs structural surface growth. In this paper we will quantify the analogies drawn above and then focus on the influence of stochastic fluctuations on structure formation. We clarify that it is already known that structure formation was predominately at superhorizon scales during the inflation era and so must be treated by general relativity [4]. However, we will show that in specific periods when dissipation becomes significant, the influence of fluctuation to structure formation is of subhorizon scales, and can be described by a nonrelativistic equation.

(1) The standard model(s) of cosmic structure formation (e.g., inflation theory) assumes that the initial spectrum of density perturbation was given by the vacuum quantum fluctuations and inflationary expansion, and that the subsequent evolution of clustering was deterministic, i.e., it obeyed a dynamical equation *without a noise term.* This is equivalent to assuming that either (a) no noise sources existed after the inflation era or (b) the influence of post-inflation noise on structure formation was negligible.

Obviously assumption (a) is not true, because dissipation (or processes of approaching locally thermal equilibrium) was essential in the eras of cosmic entropy generation, and generally such dissipative processes would lead to a stochastic force **F** (fluctuation-dissipation theorem). In the standard model, these dissipative eras at least included reheating of inflation, baryongenesis, nonthermal equilibrium decoupling of particles, and post-inflation phase transitions. Moreover, the actions given by turbulencelike perturbations and explosions were also essentially stochastic. One can expect that in such eras cosmic matter was influenced significantly by the stochastic fluctuation force $\mathbf{F}$.

Turning to assumption (b), it is correct if the nonlinear terms in the dynamical equation can be neglected. Without the nonlinear term, the noise force $\mathbf{F}$ will not change the scenario of clustering as given by the linear approximation, but only contributes to a statistical error in the result. However, the influence of noise will no longer be trivial, as we shall show, if nonlinear corrections to the dynamical equation are considered.

As an example, let us consider the era of reheating after inflation, during which there was out-of-equilibrium decay of massive, nonrelativistic particles. This process can be described as "friction"-like coupling in the dynamical equation [1]. Since by this time period causality forbids any new formation of fluctuations at superhorizon scales, the only fluctuations raised by the entropy generation of reheating are of subhorizon scales. Moreover, during the period of coherent oscillation, as the scalar field damps, the Universe becomes matter dominated by these nonrelativistic particles. Therefore, in this time interval, the influence of stochastic fluctuations on structure formation can be described by the nonrelativistic hydrodynamical equation of structure growth in an expanding universe. In linear approximation, the momentum equation is given by [3]

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{R}}{R}\mathbf{v} + \frac{\dot{R}}{R}(\mathbf{r}\cdot\nabla)\mathbf{v} + \frac{v_s^2}{\rho_0}\nabla\rho + \nabla\phi = 0 \ , \qquad (1)$$

where the density ρ, peculiar velocity $\mathbf{v}$, and gravitational potential ϕ are the perturbations to the basic-state (smooth) solutions ρ_0, $\mathbf{v_0}$, ϕ_0. $R(t)$ is the cosmic scale factor and v_s is the speed of sound. A straightforward examination of the linearized equations shows that only the vorticity free modes can be amplified by gravitational instability in an expanding universe. Therefore, we will only consider those solutions satisfying the constraint $\nabla \times \mathbf{v} = 0$. In this case, one can define a velocity potential ψ by

$$\mathbf{v} = -\nabla\psi. \qquad (2)$$

On the other hand, it is well known that in linear approximation the velocity $\mathbf{v}(\mathbf{r}, t)$ is proportional to the gravitational force produced by the surrounding density perturbation. Thus we have the local relation

$$\rho = -f\nabla\cdot\mathbf{v}, \qquad (3)$$

where $f = 4\pi\rho_0/H_0$ in a flat ($k = 0$) universe. From Eqs. (2) and (3), one has $\rho = f\nabla^2\psi$. Therefore, ψ is proportional to the gravitational potential by the relation $\phi = 4\pi G f\psi$, so that $\mathbf{v} = -(4\pi G f)^{-1}\nabla\phi$. Substituting Eqs. (2) and (3) into Eq. (1), one has

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{R}}{R}\mathbf{v} + \frac{\dot{R}}{R}(\mathbf{r}\cdot\nabla)\mathbf{v} = \frac{v_s^2}{\rho_0}f\nabla^2\mathbf{v} + 4\pi G f\mathbf{v} \ . \qquad (4)$$

This equation is similar to the Langevin equation but without a stochastic force. The first term on the right-hand side of Eq. (4) describes relaxation of the structure by diffusion. The second term formally corresponds to the viscosity term in the Langevin equation, but here the sign is negative, because self-gravitation leads to acceleration, not deceleration of the clustering matter.

As discussed above, during the eras of dissipation in the Universe, the dynamical equation (4) should include a stochastic force or noise term, $\mathbf{F}$, on the right side. Then Eq. (4) finally has the form of a Langevin-like equation. The stochastic force acting on the vorticity-free perturbation should be

$$\mathbf{F} = \nabla\eta(\mathbf{x}, t) \ , \qquad (5)$$

where the noise $\eta(\mathbf{x}, t)$ satisfies $\langle\eta(\mathbf{x}, t)\rangle = 0$. If the noise is Gaussian, we have $\langle\eta(\mathbf{x}, t)\eta(\mathbf{x}, t)\rangle = 2D\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t')$, where D is the mean square variance of the noise. More generally, the spatial-temporal Fourier transform of $\eta(\mathbf{x}, t)$ satisfies

$$\langle\eta(\mathbf{k}, \omega)\eta(\mathbf{k}', \omega')\rangle = 2Dk^{-2\chi}\omega^{-2\theta}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \ , \qquad (6)$$

where for the case of Gaussian noise $\chi = \theta = 0$.

At linear approximation as we are considering in Eq. (4), the solution will essentially not be affected by the stochastic force $\mathbf{F}$, because the noise term can be eliminated from the dynamical equation upon averaging, regardless of the value of D. As such, the noise term simply leads to an increase of statistical variance in the linear results.

However, adding nonlinear corrections to Eq. (4) will substantially change this scenario. The lowest order nonlinear correction of Eq. (4) is given by the Euler term $(\mathbf{v}\cdot\nabla)\mathbf{v}$. Including this and the noise term, Eq. (4) has the modified form

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{R}}{R}\mathbf{v} + \frac{\dot{R}}{R}(\mathbf{r}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{v_s^2}{\rho_0}f\nabla^2\mathbf{v} + 4\pi G f\mathbf{v} + \mathbf{F} \ . \qquad (7)$$

Strictly speaking, we should also include the dissipative term corresponding to $\mathbf{F}$ in Eq. (7). However, this linear term will not affect the main results discussed below.

Equation (7) governs the evolution of matter perturbation during the beginning period of reheating when the scalar field is undergoing coherent oscillations. Obviously, Eq. (7) is not limited to this period of reheating in the early Universe, but generally describes the behavior of structure formation in any era when (1) dissipation is significant, and (2) the Universe is dominated by nonrelativistic particles. Other possible examples, besides the reheating case, are late-time phase transitions [5] and collision or merging of galaxies.

If the interaction causing the stochastic force is weaker than self-gravitation, and/or its time scale is comparable to or even longer than Hubble expansion, the noise term will be less important. One can call this the case of weak noise. For instance, the stochastic force related to the bulk viscosity at last scattering surface [6] is negligible, because the entropy per baryon was very large at the era of last scattering.

However, for the eras in which the main or a comparable part of cosmic entropy was generated, as in the part of the reheating period discussed above, dissipation would have been crucial, even dominant in the evolution of the Universe [7]. Therefore the relevant stochastic force $\mathbf{F}$ would have been stronger than self-gravitation and its time scale less than that of Hubble expansion. In such periods, one can neglect the cosmic expansion $[R(t)]$ and self-gravitation $(4\pi G f \mathbf{v})$ terms. Equation (7) then becomes the Burgers' equation [8] with stochastic force

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{v_s^2}{\rho_0} f \nabla^2 \mathbf{v} + \mathbf{F} \ . \tag{8}$$

It has already been recognized that the nonlinear evolution of cosmic density inhomogeneities can be approximately treated by the Burgers' equation [9]. This work concentrated on the the formation of pancakes and filaments, and did not incorporate stochastic forces, which are central to our considerations.

Using Eqs. (3) and (8) one finds the equation for ϕ to be

$$\frac{\partial \phi}{\partial t} = f' \nabla^2 \phi + \frac{1}{2}(\nabla\phi)^2 + \eta(\mathbf{x}, t) \ , \tag{9}$$

where $f' = (v_s^2/\rho_0) f$. Both the above equations are variants of the so-called Kardar-Parisi-Zhang (KPZ) equation [2], which has been widely studied as a dynamical equation for describing universal behavior of fractal surface growth under stochastic force. From our considerations we see that the influence of stochastic forces on cosmological clustering also belongs, under certain approximations, to the dynamics given by the KPZ equation. If one makes an analogy with the theory of surface growth, one finds that the gravitational potential ϕ of cosmic matter corresponds to the height of the surface. This means that the evolution of the gravitational potential undergoing stochastic fluctuation is analogous to the problem of $d = 3 + 1$ surface growth, i.e., a "surface" growing on a three-dimensional substratum.

(2) Equations (8) and (9) show that noises in strong dissipation eras would input the corresponding scaling seeds into the mass distribution, which subsequently would then be amplified by gravitational instability in the expanding universe. As such, stochastic forces from strong dissipation eras would most likely leave some signatures in today's clustering. In order to illustrate the influence of the noise on the clustering, let us turn to the correlation functions. The seeds generated by noise normally are scaling with correlation functions going as

$$\langle \phi(\mathbf{x}, t)\phi(\mathbf{x}', t)\rangle \sim \mid \mathbf{x} - \mathbf{x}' \mid^{\alpha} \ , \tag{10}$$

so that the two-point correlation function of density is then

$$\xi(r) = \langle \rho(\mathbf{x}, t)\rho(\mathbf{x}', t)\rangle \sim r^{-\gamma} \ , \tag{11}$$

where $\gamma = 4 - \alpha$.

The index α depends on the spectrum of the noise in Eq. (6). Unfortunately, for $d = 3 + 1$, few firm relationships between $(\chi,\ \theta)$ and α are available. However, one can find the possible range of α from the following universal relation [1]:

$$\alpha = 4\beta/(\beta + 1) \ , \tag{12}$$

where β is the index for the time behavior of the correlation function of $\phi(\mathbf{x}, t)$ at a given $\mathbf{x} - \mathbf{x}'$,

$$\langle \phi(\mathbf{x}, t)\phi(\mathbf{x}', t)\rangle \sim t^{2\beta} \ . \tag{13}$$

If we examine the case of perturbations which grow faster than the gravitational instability, it would require β to be larger than $1/2$ in the radiation era or $2/3$ in the matter era. This would mean that we have, respectively,

$$0 < \gamma < 2.66 \text{ or } 2.4 \ , \tag{14}$$

thus indicating how the effects of noise, or more generally the dynamics embodied in Eq. (7), contribute nontrivially to structure formation. It also means the basic assumptions (a) and (b), implicit to the standard (inflation) model, are not necessarily true.

One can further quantify these results by using the following general relation, obtained by perturbative methods in [10],

$$\alpha = (4\chi - 2d + 8\theta + 6)/(2\theta + 3) \ , \tag{15}$$

which is valid for $0 < \chi < 2$ and $0 < \theta < 0.25$. Within the limits that one accepts this perturbative result to give a semiquantitative guide, one can obtain relations between α and the spectrum of spatial (χ) and temporal (θ) noise. For example, one can obtain the solution in the near proximity of the observed two-point density correlation function $\alpha \approx 2.3$ $(\gamma \approx 1.8)$ for $\chi \approx 2$ and $\theta \approx 1/6$. This is a suggestive example since it has the following interpretation. From Eqs. (3) and (5) observe that $\nabla^2\eta$ is proportional to the stochastic force term acting on the density fluctuation $\delta\rho$, so that this solution corresponds to a white noise fluctuation on the density but with a temporal correlation. The latter should not be surprising since dissipative eras are of finite temporal extent and so would reflect on the temporal noise correlation. The above demonstrates one way that a strong white noise in the early Universe would be able to generate an initial perturbation which along with possible further modifications by gravitational instability could leave signatures in present observation. Also contained within the solutions of (15) is $\alpha = 0$, which corresponds to the Harrison-Zeldovich spectrum. Finally, Eq. (15) tentatively shows that α increases with χ. As a reasonable extrapolation, assuming that such a trend holds for $\chi >> 2$, it is suggestive that such strong, turbulencelike noise may not be consistent with observation. Of course, we should keep in mind that Eq. (15) is a perturbative result, so that this deduction is not rigorous and also may not be unique since nonperturbative results could exist.

Standard model cosmology has eras, such as during reheating, when stochastic fluctuations given by cosmic phase transitions and other nonthermal equilibrium processes are significant. During such times, their effect may play a non-negligible role in structure formation. The last conclusion was also reached by Luo and Schramm [11] based on observational data indicating scale-free distributions of galaxy clusters. They concluded the need for incorporating a fractal structure generation mechanism into standard big-bang cosmology. Although, as pointed out by Peebles [12], a pure fractal contradicts the observed large scale angular correlation function, their considerations were restricted to subhorizon scales of order 300 Mpc in the present day Universe. They explained the mechanism based on an aggregate growth process and concluded that fractal growth originates from two-dimensional sheetlike objects. However, as we have shown, the dynamical mechanism is rather more general and is governed by the KPZ equation, which is not necessarily restrictive to two-dimensional growth phenomenon but rather to surface growth in higher dimensions also. Within the framework of the formalism presented here, we learn that the universal characteristics of rough surface growth, such as the relationship between the noise spectrum and the index of the two-point correlation function, can be used for guidance in developing models of structure formation in the Universe. This provides a two-step approach for checking models in particle cosmology: (a) testing the standard model by calculating the contributions of stochastic fluctuations related to various dissipations in the Universe; (b) testing particle physics models which may give rise to stochastic forces in the early Universe by assuming the correctness of the standard model.

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