Stability of Non-Abelian Black Holes and Catastrophe Theory

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Two types of self-gravitating particle solutions found in several theories with non-Abelian fields are smoothly connected by a family of nontrivial black holes. There exists a maximum point of the black hole entropy where the stability of solutions changes. This criterion is universal, and the changes in stability follow from a catastrophe-theoretic analysis of the potential function defined by black hole entropy.

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After Bartnik and McKinnon discovered a nontrivial particlelike structure (BM particle) in the Einstein-Yang-Mills theory [1], a variety of self-gravitating structures with non-Abelian fields have been found. Besides the BM particle, researches have discovered the colored black hole [2], the Skyrmion [3,4], or the Skyrme black hole [4–6], the monopole [7–9], or the black hole in monopole (monopole black hole) [8–10], the particle solution with massive Proca field, or the Proca black hole [11], and the sphaleron [12,11] or sphaleron black holes.

One of the most important questions about these self-gravitating non-Abelian structures is, are they stable? The BM particle and the colored black hole are unstable against radial perturbations, while both the Skyrmion and the monopole and the corresponding black hole solutions are stable. The sphaleron and its black hole solution may be unstable because of their topological structure. Are there any common properties in those non-Abelian structures? Can we find any universal understanding for them? Answering these questions is the main purpose of the present paper. We will soon show that there is a universal picture for self-gravitating non-Abelian structures that incorporates these black hole solutions, and that accounts for their stability properties via a catastrophe-theoretic analysis of the black hole entropy, with S (=the area of event horizon/4) regarded as a potential function.

We have reanalyzed five models, which are listed in Table I. Some known results concerning these models are also summarized in the table. Remarkably, except for the colored black hole and the monopole black hole, all solutions share the following properties [13]: (1) There are two particlelike solutions. One corresponds to the known particle solution without gravity [3,12], and the other has properties similar to those of the BM particle [1,4,11]. (2) Two branches of black hole solutions, which leave from two particles, bifurcate at some critical horizon radius [5,11]. Beyond this critical point, where the black hole has a maximum mass and a maximum entropy, there exists no nontrivial structure. The upper branch in Fig. 1 has larger entropy than that of the lower branch. Hence, we shall call each of them high- and low-entropy branches, respectively. The low-entropy branch is similar to the colored black hole solution, and the high-entropy branch approaches the Schwarzschild black hole in the "low energy" limit [6]. Here, low energy means that the mass of the particle is much smaller than the Planck mass $m_P \equiv G^{-1/2}$. It is realized in the limit as $\mu \to 0$, where μ is a mass of the relevant non-Abelian field, e.g., $\mu = g\Phi_0$

TABLE I. The properties of five models including non-Abelian fields. See text about the meaning of "stable" for the sphaleron black hole. BFM particle means one of two nontrivial solutions found in [9], which is more massive than the usual monopole. $C_\#$ denotes how many times the sign of the specific heat changes in the branch. The Reissner-Nordström black holes in Einstein-Maxwell and Einstein-Yang-Mills-Higgs systems are listed as references. In order to define parameters in the theories such as a gauge coupling constant g, we show the Lagrangians of non-Abelian fields and the potentials of Higgs fields.

Black holes	Non-Abelian fields	Higgs fields	Particles	Black hole modes	Mass (entropy)	$C_\#$
1. Colored BH	Yang-Mills field [SU(2)]	···	BM particle	1 (unstable)	$m < \infty$	2
	$-\frac{1}{16\pi}\mathrm{Tr}F^2, \quad F = dA + gA \wedge A$					
2. Skyrme BH	Skyrme field [SU(2)×SU(2)]	···	Skyrmion	1 (stable)	finite (high)	0
	$-\frac{1}{32g_S^2}\mathrm{Tr}F^2 - \frac{1}{4}f_S^2\mathrm{Tr}A^2, \quad F = -dA$		BM type	1 (unstable)	finite (low)	1 or 3
3. Proca BH	Massive Yang-Mills (Proca) field	···	Procaon	1 (stable)	finite (high)	0
	$-\frac{1}{4\pi}[\frac{1}{4}\mathrm{Tr}F^2 - \frac{1}{2}\mu^2\mathrm{Tr}A^2]$		BM type	1 (unstable)	finite (low)	1 or 3
4. Sphaleron BH	Yang-Mills field [SU(2)]	(complex doublet)	Sphaleron	1 ("stable")	finite (high)	0
	$-\frac{1}{16\pi}\mathrm{Tr}F^2$	$-\lambda(\Phi^\dagger\Phi - \Phi_0^2)^2$	BM type	1 (unstable)	finite (low)	1 or 3
5. Monopole BH	Yang-Mills field [SU(2)]	(real triplet)	Monopole	1 (stable)	finite (high)	0
	$-\frac{1}{16\pi}\mathrm{Tr}F^2$	$-\frac{\lambda}{4}(\Phi^2 - \Phi_0^2)^2$	[BFM particle]	[1 (unstable)]	[finite (low)]	[0]
Reissner	Electromagnetic field [U(1)]	···	···	1 (stable)	$m < \infty$	1
-Nordström BH	Yang-Mills field [SU(2)]	(real triplet)	···	1 (stable or unstable)	$m < \infty$	1

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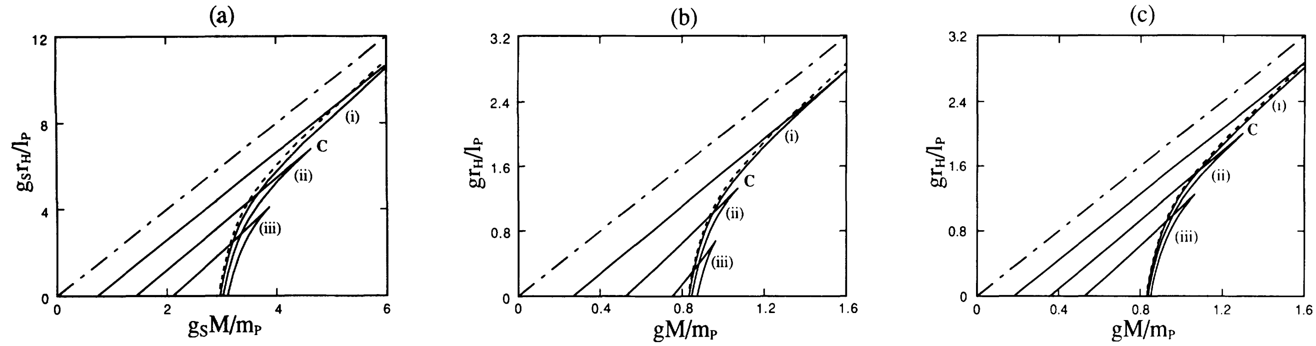

FIG. 1. The mass-horizon radius diagrams for (a) the Skyrme black hole with f_S/m_P = (i) 0.01, (ii) 0.02, and (iii) 0.03; (b) the Proca black hole with μ/gm_P= (i) 0.05, (ii) 0.10, and (iii) 0.15; and (c) the sphaleron black hole with $\lambda = 0.125$ and Φ_0/m_P= (i) 0.1, (ii) 0.2, and (iii) 0.3. C is a cusp, where the black hole has a maximum entropy. Beyond its entropy there is no non-Abelian black hole. The Schwarzschild black hole (the dot-dashed line) and the colored black hole (the dotted line) are also shown as references.

(the vacuum expectation value of the Higgs field) for the Einstein-Yang-Mills-Higgs system, or $\mu = g_S f_S$ (two coupling constants of Skyrme field) for the Einstein-Skyrme system. (Note that $g_S^2 = 4\pi g^2$ in our notation.) On the other hand, in the limit of "high energy," no solution exists. It disappears around $\mu \sim m_P/g$. (3) The high-entropy branch is stable (except for the sphaleron solution, but see below), while the low-entropy branch is unstable [4,5,14,15]. (4) The specific heat in the high-entropy branch is always negative, while the specific heat in the low-entropy branch changes its sign a few times [6].

In order to obtain a universal picture with the properties (1)–(4) above, we have reanalyzed the five models listed in Table I and found the following new results [16]: (5) Fixing the horizon radius r_H, there are two black hole solutions with different masses. Those two branches are bifurcated at some critical radius. In the mass-radius (M-r_H) plane, the solution curve has a cusp at this critical point C (see Fig. 1). The stability changes at this cusp; that is, the high-entropy branch is stable while the low-entropy one is unstable against radial perturbations. (6) If we draw the solution curve in the three dimensional space of the mass M, the entropy of the black hole S, and the field strength at the horizon $B_H \equiv (\mathrm{Tr}F^2)^{1/2}|_{\mathrm{horizon}}$, it becomes smooth (see Fig. 2). Here, the expression B_H has been used because only the radial component of the magnetic part of the non-Abelian field is finite at the horizon. Only the projection onto the M-S plane (and then onto the M-r_H plane) provides a cusp. The cusp, at which the stability changes, corresponds to a turning point in the three dimensional picture, where the black hole entropy takes the maximum value.

The appearance of a cusp as a critical point of stability is often discussed in catastrophe theory [17,18] and in its application to astrophysics [19,20]. In the present case, if we regard S, M, and B_H as a potential function, a control parameter, and a generalized coordinate, respectively, we may apply catastrophe theory to the present stability problem as follows. In catastrophe theory, solutions are regarded as extremal points on the Whitney surface, $S = S(M, B_H)$, when the control parameter M is fixed. If a solution is a maximal point, then that solution is stable because its entropy is maximal. On the other hand, if it is a minimal point, then it is unstable. At the maximum entropy, the solution turns out to be an inflection point, beyond which there is no extremal point; i.e., there is no black hole solution [16].

We may wonder what happens with the sphaleron black hole, because both its high- and low-entropy branches are unstable for topological reasons. Is this consistent with our interpretation of stability via catastrophe theory? When we discuss stability, in general there are many modes to be investigated. A general argument about the instability of the sphaleron is based on a topological analysis [13], which does not choose any specific mode. On the other hand, when we discuss stability change using catastrophe theory, we focus on some

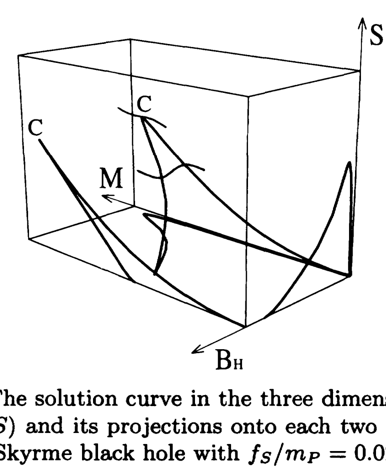

FIG. 2. The solution curve in the three dimensional space of (M, B_H, S) and its projections onto each two dimensional plane for a Skyrme black hole with $f_S/m_P = 0.02$. The cusp C in M-S plane is a critical point for stability. For the fixed control parameter M, two solutions are at extremal points on the Whitney surface; the maximal one is stable, while the minimal one is unstable. Beyond the critical point C, there is no extremal point, i.e., no non-Abelian black hole.

specific mode. For the sphaleron without gravity, the stability analysis with a spherically symmetric ansatz was done [21]. It was explicitly shown that there is only one unstable mode. Although no analysis has been made, so far, for the case of the gravitating sphaleron or the sphaleron black hole, we guess that it may be stable in the high-entropy branch against radial perturbations except for one unstable mode corresponding to the above. In the low-entropy branch, some of the stable modes become unstable. The sphaleron black hole picks up at least one more unstable mode beyond the critical point. In this sense, we argue that the high-entropy branch is "stable" while the low-entropy one is unstable. If this is so, then catastrophe theory accounts correctly for the stability of black holes even in the sphaleronic case.

From the above discussion, we see that we can classify non-Abelian black holes into two types, (A) and (B):

(A) High-entropy "neutral" types: The high-entropy branch is "stable." The field strength at the horizon (B_H) is still small as well as the black hole is globally neutral. The black hole is approximately neutral. We may adopt the following picture for this type of black hole. The non-Abelian structure may be approximated as a uniform vacuum energy density ρ_{vac} with a sphere whose radius is the Compton wavelength of the massive non-Abelian field. As for the black hole solution, the horizon must exist in the region of uniform vacuum energy. Otherwise, nontrivial non-Abelian structure is swallowed by the black hole, resulting in a trivial Schwarzschild solution. This explains why there is an upper bound on the mass or horizon radius for this nontrivial solution. From our picture, the high-entropy neutral black hole near the horizon is approximated by the Schwarzschild–de Sitter spacetime with the cosmological constant$=8\pi G\rho_{\text{vac}}$. In the limit of low energy, the solution approaches the Schwarzschild black hole. The negative specific heat is also consistent with that of the Schwarzschild or Schwarzschild–de Sitter spacetime. The self-gravitating particle approaches the known particle solution in a Minkowski background [3,12]. Such a particle can exist without gravity.

(B) Low-entropy "locally charged" types: The low-entropy branch is unstable. The structure of this type of black hole is quite similar to the colored black hole. Although the black hole is globally neutral, B_H does not vanish at the horizon. Its value is rather large. An effective charge appears near the black hole horizon. Furthermore, in the low energy limit, the solution approaches the colored black hole [4,6]. The nontrivial structure in this case is due to the kinetic term of the non-Abelian gauge field, $\text{Tr}F^2$. Gravity must play an essential role in this nontrivial structure, because the BM particle cannot exist without gravity and the mass scale is about $m_{\text{BM}} \sim m_P/g$, which is almost independent of μ (or Φ_0, f_S). Just as we found strange behavior in the specific heat of a colored black hole [6], we find changes of its sign a few times (see Table I).

As for the excited state, i.e., higher-node solutions, we find that a similar cusp exists [6,16]. We expect that when the solution goes beyond this cusp (the maximum entropy point), another instability will appear. Since the colored black hole has n unstable modes for an n-node solution [14,22], we expect that the high-entropy branch of n-node solutions has $(n-1)$ unstable modes while the low-entropy branch has n unstable modes.

We have, so far, discussed all known non-Abelian structures except for the monopole black hole and the colored black hole, and have presented a universal picture. Although the colored black hole does not always have all the properties we have discussed above, it should be included in our universal picture, because the colored black hole and the Schwarzschild black hole are obtained exactly as low energy limits of the low- and high-entropy branches, respectively. The only exceptional solution is the monopole or the monopole black hole, which is globally charged.

It should be stressed, however, that although the monopole black hole has different properties from types (A) and (B) above and shows more complicated behaviors [8–10], the catastrophe theory is again applied to the stability analysis [23]. Depending on the parameters g, λ, and Φ_0 in the Einstein-Yang-Mills-Higgs system, there seem to be the following two cases [8–10,23]:

(I) The mass of the monopole black hole increases monotonically as entropy increases and the solution eventually reaches a bifurcation point B with the Reissner-Nordström (RN) black hole branch. No cusp appears. The monopole black hole is stable, while the RN solution becomes unstable beyond this bifurcation point B [8].

(II) For some range of parameters, the solution curve of the monopole black holes has a cusp C in the M-S plane [10,23], where the black hole has the maximum entropy. There are two solutions with the same horizon radius (the same entropy) but different masses just as with the other type of nontrivial black holes. When the radius gets small in the second (low-entropy) branch, the solution either may merge to the RN black hole at a bifurcation point B or might reach another particle solution (BFM particle [9]). We guess that the second low-entropy branch is unstable while the first high-entropy branch is stable (see [10]). The RN black hole is stable before the bifurcation point B, but it becomes unstable beyond B.

All these behaviors (I) and (II), including the stability of the RN black hole, follow easily from catastrophe theory, if the entropy is regarded as the potential function [24]. The entropy S with a fixed mass M is maximal for the stable branch but becomes minimal for the unstable branch [23].

In this Letter, we have reanalyzed the known non-Abelian black holes, as well as the corresponding self-gravitating particlelike solutions. We find a universal picture: The globally neutral solutions are classified into two types depending on whether they are almost neutral or locally charged. The neutral type (high-entropy branch) is similar to the Schwarzschild–de Sitter solution and stable (see the previous discussion for sphaleron black holes). The locally charged type (low-entropy branch) is like the colored black hole and unstable. Its specific heat changes sign a few times with respect to the mass. When those two types coincide, the entropy becomes maximum. Catastrophe theory can be applied to analyze the stability of these black holes. One stable mode in the high-entropy branch becomes unstable beyond the bifurcation point. This may be true also for the sphaleron black hole. As for the globally charged black hole (monopole black hole), we can also apply catastrophe theory to the stability analysis, although the behavior of the solutions is more complicated.

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