Randomness at the Edge: Theory of Quantum Hall Transport at Filling $\nu = 2/3$

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Current Luttinger liquid edge state theories for filling $\nu = 2/3$ predict a nonuniversal Hall conductance, in disagreement with experiment. Upon inclusion of random edge tunneling we find a phase transition into a new disorder-dominated edge phase. An exact solution of the random model in this phase gives a quantized Hall conductance of 2/3 and a neutral mode propagating upstream. The presence of the neutral mode changes the predicted temperature dependence for tunneling through a point contact from $T^{2/\nu-2}$ to T^2.

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The most striking aspect of the quantum Hall effect is the remarkable quantization of the Hall conductance measured in transport experiments [1,2]. For the integer quantum Hall effect, the edge state picture formulated by Halperin [3] has been incorporated into the framework of Landauer-Buttiker transport theory [4,5], and provides an extremely simple explanation of this quantization. An edge state explanation for the quantized conductance in the fractional quantized Hall effect (FQHE), though, is more subtle. An important advance was made by Wen [6], who argued that edge excitations in the FQHE can be described at low energies by a chiral Luttinger liquid model. For the Laughlin states, such as $\nu = 1/3$, where the edge state has only one branch, this approach gives a simple explanation of the Hall quantization, and moreover has enjoyed success in describing tunneling between edge states [7,8]. However, for hierarchical quantum Hall states, such as $\nu = 2/3$, the edge state structure is much more complicated and the whole approach more problematic. Indeed, Wen [6] and MacDonald [9] have argued that a $\nu = 2/3$ edge consists of two edge branches which move in *opposite* directions. However, a recent time-domain experiment [10] has shown only *one* propagating mode at the $\nu = 2/3$ edge. Moreover, including electron interactions between the two branches in this model gives a Hall conductance which is *not* quantized and *nonuniversal*, depending on the strength of the interactions. Is something seriously amiss with the edge state approach?

In this Letter we reconsider the $\nu = 2/3$ edge and argue that the problem lies not with the edge state approach, but with the assumption of an ideal edge. In contrast to a $\nu = 1/3$ edge, where randomness is unimportant since a single chiral mode cannot backscatter, we show that for a $\nu = 2/3$ edge randomness is crucial. Specifically, we show that with random edge scattering there is an edge phase transition from the state in which the zero temperature conductance is nonuniversal to a new disorder-dominated edge phase. By exploiting a hidden SU(2) symmetry, we obtain an exact solution of the model in the disorder-dominated phase which reveals only a single propagating charge mode. This mode gives a quantized Hall conductance of 2/3. Thus, random edge scattering is apparently necessary to explain the Hall quantization at filling $\nu = 2/3$. The exact solution reveals, moreover, the presence of a *neutral* edge mode, which propagates in the direction opposite to the charge mode. The neutral mode, which does not contribute to the Hall conductance, would not be detectable in the capacitive time-domain experiments of Ref. [10]. However, we show that the neutral mode *modifies* the temperature dependence of the conductance through a point contact in a $\nu = 2/3$ fluid, changing the behavior from $G(T) \sim T^{2/\nu-2}$ to $G(T) \sim T^2$. We suggest a variant of the time-domain experiment of Ref. [10] which should allow a direct observation of the neutral mode.

McDonald and Wen [6,9] have argued that the $\nu = 2/3$ edge consists of two modes: a forward propagating edge mode, similar to a $\nu = 1$ edge, and a backward propagating mode, similar to a $\nu = 1/3$ edge. This picture is supported by microscopic calculations for a sharp edge which show a nonmonotonic electron density, increasing from $\nu = 2/3$ to $\nu = 1$ before falling to zero. Following Wen, the appropriate (Euclidean) action is

$$S_0 = \int dx\, d\tau \frac{1}{4\pi} \left[\partial_x\phi_1(i\partial_\tau + v_1\partial_x)\phi_1 + 3\partial_x\phi_2(-i\partial_\tau + v_2\partial_x)\phi_2 + 2v_{12}\partial_x\phi_1\partial_x\phi_2\right]. \quad (1)$$

When $v_{12} = 0$, this describes two independent chiral Luttinger liquids, equivalent to the edge states of the $\nu = 1$ and $\nu = 1/3$ quantum Hall fluids. The 1D electron charge density in each mode is given by $\partial_x\phi_{1,2}$. The velocities v_1 and v_2 reflect the interactions within each channel. We suppose the interactions are short ranged, screened by a ground plane. In general, there will also be a repulsive interaction $v_{12} > 0$ between the two channels. This mixes the channels together, giving eigenmodes that are linear combinations of ϕ_1 and ϕ_2. Stability requires that v_{12} is not too large, namely, $v_{12}^2 < 3v_1v_2$.

The two-terminal conductance of a $\nu = 2/3$ Hall bar with top and bottom edges described by (1) may be computed from the Kubo formula, $G = \langle I(\omega_n)I(-\omega_n)\rangle/\hbar|\omega_n|$. Here the current I is a sum of the currents on the top and bottom edges at $x = 0$, each given by $ie\partial_\tau(\phi_1+\phi_2)/2\pi$. Since (1) is quadratic, G can readily be evaluated, giving

$$G = \frac{2}{3}\Delta\frac{e^2}{h}, \qquad \Delta = (2-\sqrt{3}c)/\sqrt{1-c^2}, \tag{2}$$

with $c = (2v_{12}/\sqrt{3})(v_1+v_2)^{-1}$. The stability requirement implies that $|c| < 1$. Notice that G is nonuniversal, depending on the interaction strengths [11]. A similar calculation shows that the four-terminal Hall conductance is also nonuniversal, given by $G_H = (1/3)(1+\Delta^2)e^2/h$. There is one special choice of interactions, $c = \sqrt{3}/2$, for which both G and G_H are minimal and given by $G_{\rm min} = 2/3$. But experiments need no such fine tuning.

The absence of a quantized conductance can be traced to a lack of equilibration between the right and left moving edge channels. To see this consider the case of decoupled channels, $v_{12} = 0$, for which $G = 4/3$, rather than 2/3. The value 4/3 also follows from a simple Landauer type argument, in which current from the source injected into the ϕ_1 mode on, say, the top edge of a Hall bar is added to another current injected from the source into the ϕ_2 mode on the *bottom* edge. Clearly, this requires that the two opposite moving modes on a given edge do *not* equilibrate with one another. To allow for possible equilibration, the interaction v_{12} is also insufficient. Rather, it is essential to allow the electrons to tunnel between the two edge modes. However, a (spatially) constant intermode tunneling term is ineffective at backscattering, since the two modes will generally have different momenta. (The momentum difference, which is gauge invariant, is proportional to the magnetic flux penetrating between the two edge modes.) But if the intermode tunneling is (spatially) *random*, momentum along the edge is not conserved, and backscattering (and hence equilibration) is possible.

Consider then random electron tunneling between the two modes ϕ_1 and ϕ_2. The operator $\exp(i\phi_1)$ adds an electron to the mode ϕ_1, whereas $\exp(-i\phi_2)$ adds a 1/3 charge Laughlin quasiparticle to the mode ϕ_2. Since interedge tunneling conserves charge, the simplest (and most relevant) term is $\exp(i\phi_1+3i\phi_2)$, which hops three quasiparticles, with total charge e, from mode 2 to mode 1. The corresponding term to add to the action (1) is thus

$$S_1 = \int dx\, d\tau\, \xi(x)\exp(i\phi_1+3i\phi_2) + \text{c.c.} \tag{3}$$

In general ξ is complex. Indeed, for (spatially) uniform tunneling one would have $\xi = t\exp(ik_{12}x)$ with k_{12} the momentum difference between the two modes. But since this oscillates rapidly as x varies it is unimportant at long length scales. For a realistic edge with impurities, $\xi(x)$ will be random and uncorrelated at large separations. For simplicity we assume that $\xi(x)$ is a Gaussian random variable satisfying $\overline{\xi^*(x)\xi(x')} = W\delta(x-x')$.

For small randomness W the effect of the tunneling term can be analyzed by evaluating the pair correlation function of the tunneling operator $O = \exp(i\phi_1+3i\phi_2)$ using the quadratic action (1). We find

$$\langle O(x,\tau)O(0,0)\rangle_0 \sim \frac{1}{(v_+\tau+ix)^{\Delta+1}}\frac{1}{(v_-\tau-ix)^{\Delta-1}}, \tag{4}$$

with Δ given in (2). The total scaling dimension of the operator, a sum of right and left moving dimensions, is equal to Δ. The difference between the dimensions is 2, consistent with boson statistics of the operator O.

Under a renormalization group (RG) transformation which leaves the quadratic action (1) invariant, the dimension Δ determines the relevancy of O. Since the perturbation (3) is spatially random [12], the leading RG flow equation for W is

$$\frac{dW}{d\ell} = (3-2\Delta)W. \tag{5}$$

For $\Delta > 3/2$, which corresponds to small interaction v_{12}, small randomness is irrelevant. In this case at $T = 0$, the conductance $G = 2\Delta/3$ is nonuniversal [13], with $G > 1$. At the transition $G^* = 1$. For $\Delta < 3/2$, weak random tunneling grows and drives the edge into a disorder-dominated phase.

Can this disorder-dominated phase correctly describe a real $\nu = 2/3$ edge with quantized Hall conductance? Clearly perturbation theory in W is useless, and a nonperturbative approach is necessary. Since our model is a chiral generalization of an interacting 1D localization problem, this might seem hopeless. However, we now show that the strong coupling phase has a hidden SU(2) symmetry, which allows for a complete solution. To this end introduce first new fields:

$$\phi_\rho = \sqrt{3/2}(\phi_1+\phi_2), \qquad \phi_\sigma = \sqrt{1/2}(\phi_1+3\phi_2). \tag{6}$$

Here ϕ_ρ is a "charge" mode and ϕ_σ is a "neutral" mode. The charge current along an edge is $i\partial_\tau\phi_\rho/2\pi$. The total action $S = S_0 + S_1$ can be reexpressed as $S = S_\rho + S_\sigma + S_{\rm pert}$ with charge and neutral pieces

$$S_\rho = \frac{1}{4\pi}\int_{x,\tau}\partial_x\phi_\rho(i\partial_\tau+v_\rho\partial_x)\phi_\rho, \tag{7}$$

$$S_\sigma = \frac{1}{4\pi}\int_{x,\tau}\partial_x\phi_\sigma(-i\partial_\tau+v_\sigma\partial_x)\phi_\sigma + \int_{x,\tau}[\xi(x)e^{i\sqrt{2}\phi_\sigma} + \text{c.c.}], \tag{8}$$

coupled together via

$$S_{\text{pert}} = v\frac{1}{4\pi}\int_{x,\tau} \partial_x\phi_\rho\partial_x\phi_\sigma. \qquad (9)$$

The velocities v_ρ, v_σ, and v depend on the original velocities in (1) in a complicated way (which we do not display), but for $\Delta - 1$ small are simply related to one another: $v = (v_\rho + v_\sigma)\sqrt{(\Delta - 1)/2}$. Thus at $\Delta = 1$, the interaction v vanishes, and the charge and neutral modes decouple. As we now show, this decoupled problem can be solved exactly for arbitrary randomness W, giving a fixed line at $\Delta = 1$ in the Δ-W plane (see Fig. 1). The coupling term S_{pert} is then shown to be an irrelevant perturbation, so that the RG flows are towards the decoupled fixed line.

When $v = 0$ the two operators $\cos(\sqrt{2}\phi_\sigma)$ and $\sin(\sqrt{2}\phi_\sigma)$ in S_σ both have scaling dimension $\Delta = 1$, as does $\partial_x\phi_\sigma$. These three operators together satisfy an SU(2) algebra, known as a level-one SU(2) current algebra. The quadratic part of the action S_σ respects this SU(2) symmetry, whereas the nonlinear terms are random SU(2) symmetry breaking fields. This SU(2) symmetry allows us to write down a model in terms of a two component fermion ψ, which has identical low-energy physics to S_σ. We first introduce an *extra* bosonic field χ with free action S_χ the same as that of ϕ_σ in the first term of Eq. (8). The action $S_\sigma + S_\chi$ is then equivalent to the bosonized representation of a model of spin 1/2 chiral fermions,

$$S_\psi = \int_{x,\tau} \psi^\dagger(\partial_\tau + iv_\sigma\partial_x)\psi + \psi^\dagger(\xi\sigma^+ + \xi^*\sigma^-)\psi, \quad (10)$$

provided we identify $\psi_1 = \exp[i(\chi + \phi_\sigma)/\sqrt{2}]$ and $\psi_2 = \exp[i(\chi - \phi_\sigma)/\sqrt{2}]$. Here $\sigma^\pm = \sigma_x \pm i\sigma_y$ with Pauli matrices σ_μ. The operator corresponding to $\partial_x\phi_\sigma$ is $\psi^\dagger\sigma_z\psi$. Note that χ does not enter physical observables, but allows a convenient representation of the SU(2) symmetry.

The random terms can now be completely eliminated from the action (10) by performing a unitary SU(2) gauge transformation, $\tilde{\psi}(x) = U(x)\psi(x)$, with

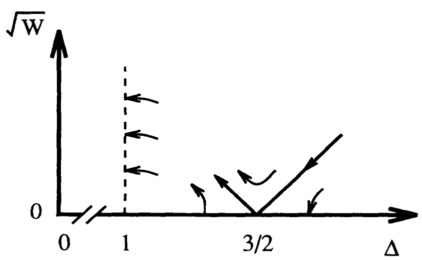

FIG. 1. Renormalization group flow diagram as a function of disorder strength W and the scaling dimension Δ of the tunneling operator. For $\Delta < 3/2$ all flows end up at the exactly soluble fixed line $\Delta = 1$.

$$U(x) = T_x \exp\left[-i\int_{-\infty}^{x} dx' M(x')\right], \qquad (11)$$

where the 2×2 matrix $M(x) = [\xi(x)\sigma^+ + \xi^*(x)\sigma^-]/v_\sigma$ and T_x is an x-ordering operator. In terms of the new field the action is that for free (chiral) fermions:

$$S_{\tilde{\psi}} = \int_{x,\tau} \tilde{\psi}^\dagger(\partial_\tau + iv_\sigma\partial_x)\tilde{\psi}. \qquad (12)$$

Taken together, (7) and (12) represent a complete solution of the decoupled line, $\Delta = 1$. Since both actions are quadratic, the line is in fact a *fixed line*. Notice that $S_{\tilde{\psi}}$ describes a *propagating* neutral mode. The model does *not* exhibit localization. The original neutral mode, ϕ_σ in (8), is not conserved with randomness and does not propagate.

Consider now S_{pert} which couples together the neutral and charge sectors. In terms of $\tilde{\psi}$ this becomes

$$S_{\text{pert}} = v\frac{1}{4\pi}\int_{x,\tau} \partial_x\phi_\rho\tilde{\psi}^\dagger U(x)\sigma_z U^\dagger(x)\tilde{\psi}. \qquad (13)$$

The relevancy of this term on the fixed line described by $S_\rho + S_{\tilde{\psi}}$ can be obtained from the scaling dimension of the operator $\hat{O}_v = \partial_x\phi_\rho\tilde{\psi}^\dagger\tilde{\psi}$, which we denote δ_v. Using (7) and (12) one readily obtains $\delta_v = 2$. This implies that a spatially uniform coefficient of $\hat{O}_v$ would be marginal. However, $\hat{O}_v$ has a random x-dependent coefficient, $\tilde{v}(x) = v(U\sigma_z U^\dagger)$, which is uncorrelated on spatial scales large compared to v_σ^2/W. We thus analyze the linear RG flow equation for the mean square average $W_v \equiv \langle\tilde{v}^2\rangle$ [12], which is $\partial W_v/\partial\ell = (3 - 2\delta_v)W_v$. W_v is irrelevant, reflecting the fact that the mean square average of $\tilde{v}(x)$ over a length scale $L \gg v_\sigma^2/W$ goes to zero as $v^2(v_\sigma^2/WL)$. Since $W_v \sim v^2 \sim \Delta - 1$, the RG flows are towards the fixed line at $\Delta = 1$ (see Fig. 1).

While we cannot ascertain the precise domain of attraction of the $\Delta = 1$ fixed line, some information can be obtained by performing a perturbative RG near the phase transition, when W and $\Delta - 3/2$ are small. The renormalization of W is described by (5). In terms of the right and left moving velocities, $v_\pm$ in (4), the remaining RG equations linear in W are

$$\frac{d\Delta}{d\ell} = -8\pi\frac{\sqrt{v_+v_-^{-3}}}{v_+ + v_-}(\Delta^2 - 1)W, \qquad (14)$$

$$\frac{dv_\pm}{d\ell} = -4\pi\frac{v_\pm^2}{\sqrt{v_+v_-^5}}(\Delta \mp 1)W. \qquad (15)$$

Note that the transition has a Kosterlitz-Thouless (KT) type form, with Δ being driven downward for $W \neq 0$ (see Fig. 1). In view of this it is most plausible that all of the flows above the KT sepatrix eventually make their way to the $\Delta = 1$ fixed line where the charge and neutral sectors decouple and the conductance is quantized, $G = 2/3$.

While the SU(2) neutral mode is conserved at $T = 0$ on the ($\Delta = 1$) fixed line, at finite temperatures the presence of the irrelevant operator v in (13) destroys this conservation and leads to a finite lifetime $1/\tau_\sigma \propto v^2T^2/W$. The charge mode is conserved and does not decay. However, its dispersion is modified $\omega = v_\rho q + iDq^2$, with $D \propto v^2/W$, independent of temperature, which implies a diffusive spreading of a charge pulse. It is noteworthy that we find no correction to the quantized dc conductance due to nonzero v [13].

Despite carrying no charge, the upstream propagating neutral mode can be detected in at least two ways. The first, which is less direct, involves tunneling through a constricted point contact in a $\nu = 2/3$ Hall fluid. For filling ν^{-1} an odd integer, where the edge mode has only one branch, it was shown that the conductance through a point contact vanishes with temperature as $G(T) \sim T^{2/\nu-2}$. For $\nu = 2/3$ one might therefore expect a power law, $G(T) \sim T$. However, the presence of the neutral mode at the $\nu = 2/3$ edge increases this power by 1, giving the prediction $G(T) \sim T^2$, as we now show.

The most general edge tunneling operator at a $\nu = 2/3$ edge is $O_{n_1,n_2} = e^{i(n_1\phi_1+n_2\phi_2)}$, with integers n_1 and n_2. This operator creates an edge excitation with charge $Q(n_1,n_2) = e(n_1 - n_2/3)$. For tunneling through a point contact the important quantity is the "local" dimension, δ, of the operator, $\delta(n_1,n_2)$, defined via $\langle O(x,\tau)O(x,0)\rangle \sim \tau^{-2\delta}$. Being at one spatial point, this average is independent of the random SU(2) gauge transformation (11), and δ can be readily evaluated using the free action (7) and (12). We find

$$\delta(n_1,n_2) = \frac{3}{4}\left(n_1 - \frac{n_2}{3}\right)^2 + \frac{1}{4}(n_1 - n_2)^2, \qquad (16)$$

where the first and second contributions are from the charge and neutral sectors, respectively. Let t be the amplitude for tunneling an electron through the point contact. The RG equation for t is $\partial t/\partial \ell = (1-2\delta)t$, with δ the dimension of the charge e operator which minimizes (16), namely, $\delta = \delta(1,0) = 1$. This gives an effective temperature-dependent amplitude, $t_{\text{eff}}(T) \sim tT^{2\delta-1}$, and a conductance which varies as $G(T) \sim t_{\text{eff}}^2 \sim t^2T^2$. If the neutral mode were absent, one would have $\delta = 3/4$, and hence $G(T) \sim T^1$. Thus, a measured T^2 temperature dependence would give (indirect) evidence of the neutral mode.

Time-domain transport through a large quantum dot at filling $\nu = 2/3$ might enable a much more direct measurement of the neutral mode. Imagine two leads coupled via tunnel junctions to opposite sides of a dot. A short current pulse incident in one lead, upon tunneling into the dot, would excite both the charge and neutral edge modes. These excitations, after propagating along the edge of the dot in opposite directions and at different velocities, would, upon arrival at the far tunnel junction, excite *two* current pulses into the outgoing lead. By tailoring the placement of the leads, a measurement of the direction of propagation and decay length of the neutral mode should also be possible.

In conclusion, we have solved exactly a model of a disordered $\nu = 2/3$ edge. This solution enables an analysis of resonant tunneling and nonequilibrium transport. This analysis, and a generalization to other composite FQHE edges will be reported elsewhere.

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