Anomalous Approach to the Self-Organized Critical State in a Model for "Life at the Edge of Chaos"

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(Received 15 November 1993)

A recently proposed model for the coevolution of biological species, known to exhibit self-organized criticality, is shown to have a dynamics which can be characterized by static critical exponents in the universality class of directed percolation. The dynamics does not exhibit the usual critical slowing down typically observed in second order phase transitions. The dynamics also modifies the critical distribution of avalanches. An exact solution for the mean field approximation of the model, along with scaling arguments and simulation results, show that these novel properties result from the use of critical fluctuations by the dynamics to drive the system to criticality.

PACS numbers: 87.10.+e, 02.50.−r, 05.40.+j, 05.70.Jk

The observation that a wide range of nonequilibrium systems evolve towards a critical state, without any external fine tuning of their parameters, has opened up many new areas of research. The hope that a general theory exists governing the statistical behavior of these systems has prompted some to study simple prototypical models exhibiting what many now call "self-organized criticality" [1-3].

On the other hand, theoretical biologists have promoted the idea that the evolution of species coexisting in an ecosystem is best described as a system driven to the "edge of chaos." This is based on the suggestion put forward by Kauffman and co-workers [4] that, in the steady state, an ecosystem is balanced just at the point where a change in the average probability for survival in the population of a single species can affect most all of the other species.

If this were a true description of a mature ecosystem, then the steady state would be a stable attractor, where the resulting larger correlations between the changing populations of species would make the system critical in the sense of a second order phase transition. The ability of the system to spontaneously drive itself into this type of steady state would make it, at least qualitatively, an example of self-organized criticality. This line of reasoning has recently led Bak and Sneppen to suggest a simple model for the coevolution of species which exhibits self-organized criticality [5].

The model is defined in general dimension as follows. Each species exists at a vertex on a d dimensional hypercubic lattice. A measure of the distance between two particular species in the lattice represents their relative dependence upon one another. For example, two vertices representing a predator and its prey are adjacent, because a change in either of their populations directly influences the other. Each vertex is labeled with the survival probability, x, or the corresponding species. The survival probability is the key variable in the model. It inversely measures the susceptibility of the species to change its genetics, in an unspecified manner.

The system is updated by searching it for the lattice site having the lowest survival probability, and then changing the survival probability by selecting a new value from a uniform distribution. This is meant to model either an adaptation of the weakest species as a result of selective pressure, or the filling of the ecological niche by another species. The nearest neighbors of the newly changed site are then relabeled with new survival probabilities in the same manner. This step models the adjustment of the species most strongly dependent upon the initially selected site. The entire procedure is repeated until the system reaches its steady state.

The present Letter will not attempt to justify the model as a valid description for the coevolution of species; the interested reader is referred to previous papers [5,6]. Instead, the purpose of this work is to study the properties of the transient and steady state dynamics. The model is rich enough to display many surprising features which are ostensibly quite characteristic of self-organized criticality phenomena appearing in a great number of nonequilibrium systems. However, it is simple enough to allow a quantitative understanding of its dynamics, which are intractable in other models. The properties of the dynamics are radically different from what is ordinarily observed at an equilibrium second order phase transition.

One of the most important features of the dynamics is that it does not exhibit critical slowing down. Usually, when a system is equilibrating near a critical point, macroscopic quantities decay asymptotically according to an exponential time dependence $\exp(-t/\tau)$. Here, t is the time normalized by the number of elements in the system. The longest decay mode τ is a function of the correlation length ξ, and follows the relation $\tau \sim \xi^z$, where z is the (positive) dynamic critical exponent. For the coevolution model, the elements are equivalent to species. It will be shown that, above the upper critical dimension $d=4$, the decay to the steady state is exponential, but τ is const ($z=0$). For dimensions $d<4$, the system decays according to a power law.

Another surprising feature is that the model is inti-

0031-9007/94/72(25)/4045(4)$06.00

mately related to directed percolation. The static critical exponents in directed percolation determine the dynamic exponents in the coevolution model. On the other hand, the dynamics influences the singular behavior of the time averaged quantities, so that their critical exponents are generally not given directly by the directed percolation exponents, but rather by combinations of them.

This Letter will elucidate the interesting features in the dynamics and show how the model naturally embodies them without the introduction of new dynamic critical exponents. The transient dynamics will first be solved in a mean field approximation of the model. The results will be used to construct a scaling argument for general dimension, giving the asymptotic time dependence. Finally, the influence of the critical steady state dynamics on the time average quantities will be shown. All of the analytic results are backed up by Monte Carlo simulations.

An exactly solvable mean field approximation of the model can be defined by modifying the updating procedure. The site with the lowest survival probability is selected and changed as before, but now k "neighboring" sites are chosen at random from the remaining system. Their survival probability is then changed, instead of the nearest neighbors of the initial site. The model now effectively has an infinite range interaction between sites, but only k sites are chosen instead of the entire system, in which case the model would immediately reach a trivial steady state.

Consider the distribution of sites $S(x)$ as a function of their survival probabilities x. The updating procedure biases the average survival probability, $\bar{x}$, to increase with time. Because new x are always drawn from a uniform distribution, the site distribution $S(x)$ should resemble a piecewise continuous sequence of constant functions at any given time. This observation is not restricted to the mean field approximation, but is in fact true in general dimension. Figure 1 shows simulation data for $S(x)$ in $d=1$, 2, 3, and 4 dimensional models in the critical steady state. There appears to be only one discontinuity in the distribution at the value x_c, where, for $x_c < x < 1$, $S(x)$ is much larger than for $0 < x < x_c$. The simulations evolve from an initial state where the site distribution is uniform over the entire interval $0 < x < 1$. At times prior to the steady state, the site distribution $S(x)$ has the same form as those in Fig. 1, except that the discontinuity appears at a value x_m, where $x_m < x_c$. As time increases, $x_m \rightarrow x_c$.

The mean field approximation of the model may be solved by considering the time evolution of the location of the discontinuity $x_m(t)$ in the site distribution, discretized in intervals Δx. Let $\mathcal{N}$ represent the number of discrete intervals in the distribution so that $\mathcal{N}\Delta x = 1$. Furthermore, let N be the total number of sites (species) in the system. The first objective is to calculate the time Δt it takes x_m to increase by Δx.

The probability $p(n)$ that a single site is assigned a survival probability x, with $x_m < x$, in n updates is

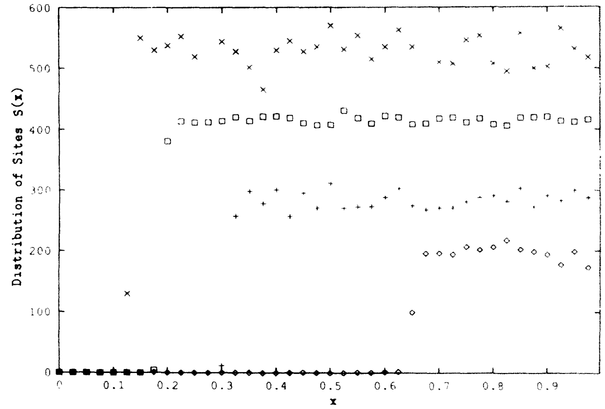

FIG. 1. Simulation data showing the distribution of sites $S(x)$ having survival probabilities x for systems in the steady state. Dimensions $d=1$, 2, 3, and 4 are shown with diamonds, plusses, squares, and crosses.

$$p(n) = (1 - x_m) x_m^{(n-1)} . \tag{1}$$

The average number of updates $\bar{n}$ for the site to have x where $x_m < x$ is therefore

$$\bar{n} = \sum_{n=1}^{\infty} n p(n) = \frac{1}{1 - x_m} . \tag{2}$$

The total number of sites which must have their survival probabilities changed in order to increase x_m to $x_m + \Delta x$ is the height H of the distribution $S(x)$ when $x_m < x$:

$$H = \frac{N \Delta x}{1 - x_m} . \tag{3}$$

Here, N is assumed to be suitably large so that fluctuations in the height of the distribution can be neglected.

Now the effect of randomly selecting k sites at each update will be considered. This will increase the time interval Δt by a large amount, and it is ultimately the source of the singular behavior at criticality. Of the k selected sites, kx_m will, on average, be given new survival probabilities x with $x < x_m$. The average number of updates required to assign all of these sites a value x with $x_m < x$ is $\bar{n}kx_m$, where $\bar{n}$ is given by Eq. (2). Finally, the fact that the initial site takes an average of $\bar{n}$ updates to have a survival probability $x_m < x$ causes the time to increase by a factor of $\bar{n}$ to $\bar{n}^2 k x_m$.

Of course, the kx_m sites will give rise to a second "generation" of sites to update. In general, an infinite number of generations are possible, with the average time to make all of the survival probabilities greater than x_m of the ith one being $\bar{n}^{i+1} k^i x_m^i$. The total time T for this cascade effect to occur, if it is finite, is given by the geometric sum:

$$T = \sum_{i=0}^{\infty} \bar{n}^{i+1} k^i x_m^i = \frac{\bar{n}}{1 - k x_m \bar{n}} . \tag{4}$$

The cascade, initiated from the update of a single site, is analogous to the avalanches in other models exhibiting

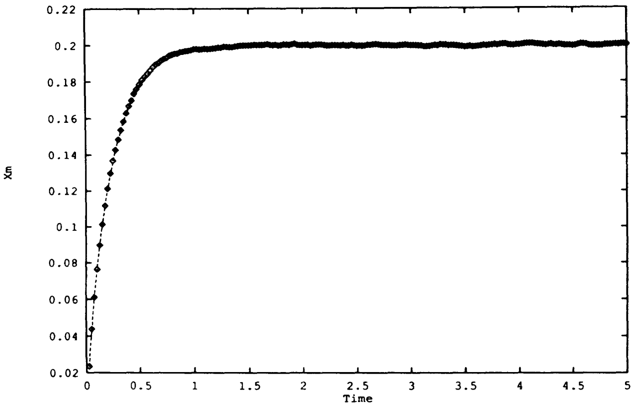

FIG. 2. Time evolution of the jump survival probability x_m for a mean field system with $k=4$. The diamonds are Monte Carlo data, and the dashed line is a plot of Eq. (7).

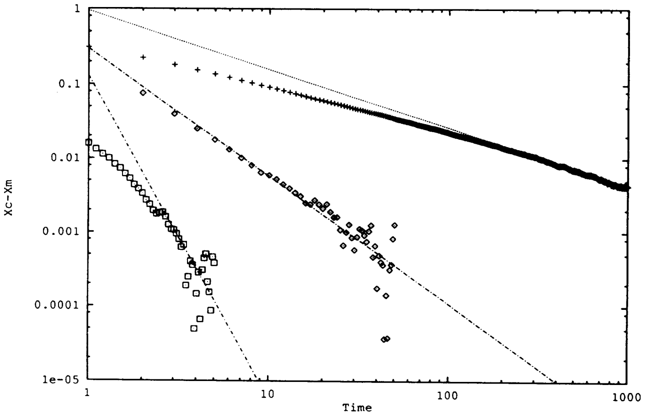

FIG. 3. Time evolution of the reduced jump survival probability, $x_c - x_m$. Plusses, diamonds, and squares are Monte Carlo data for $d=1$, 2, and 3, respectively. The dashed lines are fits to Eq. (9) using $d+1$ directed percolation exponents.

self-organized criticality (e.g., sandpiles).

The total time for the discontinuity in the distribution to move from x_m to $x_m + \Delta x$ is given by the product of Eqs. (3) and (4) [using Eq. (2)]:

$$\Delta t = \frac{N \Delta x}{(1-x_m)[1-(k+1)x_m]} . \tag{5}$$

The time it takes x_m to reach a given value, starting from an initially uniform distribution with $x_m(0)=0$, is the sum over Δt up to the desired value of x_m. Taking the continuum limit, one obtains

$$t = \int_0^{x_m(t)} \frac{dx'_m}{(1-x'_m)[1-(k+1)x'_m]} , \tag{6}$$

where t is normalized by the size of the system, N. Performing the integration and inverting give the final result:

$$x_m = \frac{e^{kt}-1}{(k+1)e^{kt}-1} . \tag{7}$$

This last equation shows that the x_m approaches the value $x_c = 1/(k+1)$ for large t. Upon reflection, it is clear that the mean field approximation describes a branching process with coordination number $k+2$, as was observed by Flyvbjerg, Sneppen, and Bak [6].

Equation (7) also shows that there is no critical slowing down in the conventional sense. The system decays to criticality as $\exp(-kt)$, where the time constant $\tau = 1/k$ is a fixed parameter of the system, independent of a correlation length.

Figure 2 shows the time evolution of a mean field system with $k=4$. The dashed line is a theoretical plot obtained from Eq. (7). The diamonds are the Monte Carlo data from the average of 10^2 simulations with $n=10^4$. The value of x_m was calculated from the measured average survival probability $\bar{x}$, determined from the relation $x_m = 2\bar{x} - 1$. The agreement between the two plots shows that the analytic theory is correct within the resolution of the simulation data.

The mean field analysis gives powerful insight into the behavior of the model on a d dimensional lattice. The key feature driving the critical behavior is the divergence of the size of the avalanches as $x_m \to x_c$. Furthermore, the singular behavior determining the time Δt for $x_m \to x_m + \Delta x$ is the average size of the avalanches. It should therefore be true that, in general, the asymptotic scaling in the transient dynamics can be obtained from a form analogous to Eq. (6):

$$t \sim \int^{x_m(t)} (x_c - x'_m)^{-\gamma} dx'_m \tag{8}$$

so that

$$x_m(t) \sim x_c - c_0 t^{1/(1-\gamma)} . \tag{9}$$

Here, γ is the mean cluster size exponent for a given class of percolation problems.

The d dimensional model appears to bear a strong resemblance to $d+1$ directed percolation. Assuming that these two models are in the same universality class allows one to calculate the scaling of the transient dynamics immediately. Substituting in the $d+1$ mean cluster size exponent for directed percolation into Eq. (9) gives a theoretical prediction. Figure 3 shows data from 1, 2, and 3 dimensional simulations of the model, where the fitting parameter is the value of x_c. The data show reasonable agreement with the theory, as indicated by the dashed lines [7].

In the critical steady state, the model gives interesting behavior in terms of the distribution of the number of avalanches $C(s)$ as a function of their size s. Following the same line of reasoning in the scaling theory for the transient dynamics, one would expect $C(s)$ to be analogous to the cluster size distribution in $d+1$ directed percolation, and that $C(s) \sim s^{1-\tau}$, where τ is the cluster size distribution exponent for directed percolation.

However, the steady state dynamics act to change the

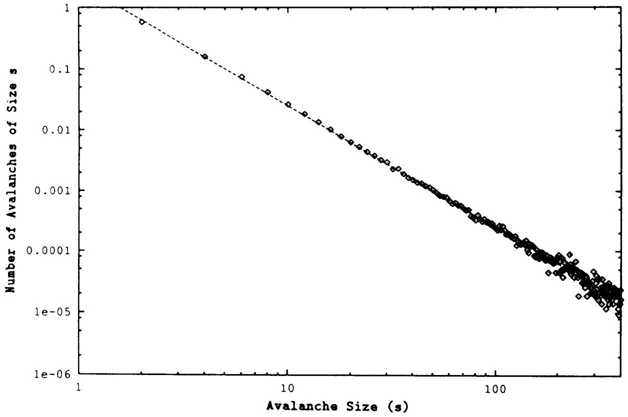

FIG. 4. Avalanche size distribution for a $d=1$ system at the steady state. The dashed line is a fit to Eq. (11) using 1+1 directed percolation exponents.

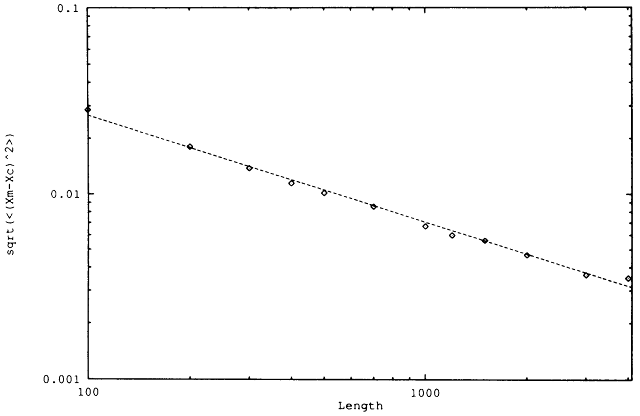

FIG. 5. Finite-size scaling measurement of the correlation exponent ν of the $d=1$ model. The dashed line is a fit to $L^{-1/\nu_{\parallel}}$ for 1+1 directed percolation.

value of the exponent. The critical state is inherently nonequilibrium, and this leads to strong fluctuations of the system near x_c. At a given value of the discontinuity x_m, the cluster distribution obeys the following relation according to standard scaling theory:

$$C(s;x_m) \sim s^{(1-\tau)} f(s|x_c - x_m|^{1/\sigma}) , \qquad (10)$$

where $f(y)$ is a scaling function which cuts off sharply for large y. The fluctuations about x_c can be included by integrating Eq. (10) near x_c. This results in the modified scaling form

$$C(s) \sim s^{1-\tau-\sigma} . \qquad (11)$$

This is very similar to ideas recently suggested by Sornette [8] as to how critical exponents are modified in nonequilibrium systems by "sweeping" through the critical point. Figure 4 shows a comparison between the avalanche distribution for a $d=1$ simulation of the coevolution model as compared to the theoretical prediction from Eq. (11), using directed percolation exponents in 1+1 dimensions. They appear to be in excellent agreement.

On the other hand, the dynamics does not modify certain exponents. For example, a measurement from a $d=1$ simulation of the correlation length exponent ν is shown in Fig. 5. The finite-size scaling method of Reynolds, Stanley, and Klein [9] has been extended to the present model by measuring the steady state fluctuations in x_c as a function of the linear dimension of the system, L [10]. The fluctuations scale as $L^{-1/\nu}$. The dashed line is the value of $1/\nu_{\parallel}$ (≈ 0.576) for 1+1 directed percolation, which agrees well with the data.

In conclusion, the model advanced by Bak and Sneppen [5] for the coevolution of species appears to reveal many interesting properties in its dynamics, which are very likely general features of many systems exhibiting self-organized criticality. The lack of conventional critical slowing down is particularly surprising. The mean field dynamics decays quickly with an exponential form having a nondiverging time constant. In finite dimensions, below $d=4$ (the upper critical dimension for $d+1$ percolation), the decay is slower than exponential. The critical fluctuations seem to aid the decay of the system to the critical state in mean field, but severely inhibit it in lower dimensions.

It is hoped that some of the ideas and techniques developed here can be used to understand the dynamics in other systems exhibiting self-organized criticality.

The authors would like to thank P. Bak, D. Stauffer, E. Lage, V. Privman, L. Schulman, and C. Doering for many useful comments and criticisms.

Note added.—While this manuscript was in preparation, the authors received a preprint of Pazcuski, Maslov, and Bak in which Eq. (9) for the transient dynamics was derived, and the relation to directed percolation was suggested using field theoretic arguments [11].

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