Critical Behavior for Correlated Strongly Coupled Boson Systems in 1 + 1 Dimensions

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The natural integrable *correlated* strongly coupled boson system in 1 + 1 dimensions is the q-boson hopping model; we calculate its critical exponent θ and determine its correlation functions. For small couplings the q-boson model has natural connections with the Bose gas and the XY models of very large spin for which θ's and correlators are reported. For large couplings the hopping model is a new phase of interacting bosons substantially different from the impenetrable Bose gas.

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The discovery of high-T_c superconductivity has stimulated studies, e.g., Refs. [1–4], of electron lattice models whose kinetic energies involve electron *correlation,* i.e., the hopping terms between adjacent sites depend on the occupation of those sites. Typically, as in, e.g., Refs. [1–3], they have involved hopping terms taken in the form

$$\mathbf{H}_{\mathrm{kin}} = \sum_{\langle j,k\rangle}\sum_{\sigma=\pm1} \mathbf{P}_{j,-\sigma} c^{\dagger}_{j,\sigma} c_{k,\sigma} \mathbf{P}_{k,-\sigma} + \mathrm{H.c.} \qquad (1)$$

The $c^{\dagger}_{j,\sigma}, c_{k,\sigma}$ (for fermions) are anticommuting spin-$\frac{1}{2}$ operators, $\{c_{j,\sigma} c^{\dagger}_{k,\sigma'}\} = \delta_{j,k}\delta_{\sigma,\sigma'}$, $\mathbf{P}_{j,\sigma} = (1 - \hat{n}_{j,\sigma})$ with $\hat{n}_{j,\sigma} \equiv c^{\dagger}_{j,\sigma} c_{j,\sigma}$, and these prevent two electrons from appearing on the same site. In this Letter we report the construction of certain comparable strongly coupled correlated *boson* models.

In Ref. [3] the strongly coupled fermion models are extended so as to combine in the one model both the Hubbard and the *t-J* models: Typically, each of these different models exhibits superconductivity induced by purely electron-electron interactions. A feature of the Hubbard model [5], of the supersymmetric *t-J* models [3], their extensions [3], and indeed the model of Ref. [4] is that in one dimension (1D models), they are completely integrable, solved by the Bethe ansatz (BA).

These high-T_c studies have also led to the investigation of various strongly coupled boson models; but, e.g., for the strongly interacting $d = 1$ boson systems studied in Ref. [6] in analogy with the fermion Hubbard models, these models are *not* solvable by the BA method [7]—essentially because of the many particles possible on a lattice site in these boson cases. However, the natural $d = 1$ boson models analogous to the correlated fermion models are not the strongly coupled boson models of Refs. [6, 7]. Instead they are the *q-boson* models, like the "q-boson hopping model" and its equivalents, reported and solved very recently [8].

To see this observe that, for (spinless) boson operators $b_j, b_j^{\dagger}$ on sites j, $[b_j, b_k^{\dagger}] = \delta_{jk}$ with $N_j = b_j^{\dagger} b_j$, and a boson projector $\mathbf{P}_j$ should eliminate boson hoppings to and from fully occupied sites j—so it should have value zero for the boson occupation number $n_j \to \infty$. It should be finite for $n_j = 0$ and unity for weak enough coupling.

Obvious possible choices for $\mathbf{P}_j$ are now $\mathbf{P}_j = ([N_j]/N_j)^{\alpha}$ with $\alpha \in \mathbf{R}$ and $[N_j] \equiv (1 - q^{-2N_j})/(1 - q^{-2})$; $q = e^{\gamma}$, $\gamma > 0$; then γ will prove to be the coupling constant of the theory. However, for integrability when $d = 1$ we find we must use $\alpha = \frac{1}{2}$. With this choice the kinetic energy of the natural correlated boson model is $\mathbf{H}_{\mathrm{kin}} = \sum_{\langle j,k\rangle} \mathbf{P}_j b_j^{\dagger} b_k \mathbf{P}_k$ + H.c., and if we *define*

$$B_j = (B_j^{\dagger})^{\dagger} = b_j \mathbf{P}_j, \qquad N_j = b_j^{\dagger} b_j, \qquad (2)$$

we find the closed algebra

$$[B_j, B_k^{\dagger}] = q^{-2N_j}\delta_{jk}, \qquad [N_j, B_k^{\dagger}] = B_j^{\dagger}\delta_{jk}, \quad \text{and} \quad \mathrm{H.c.} \qquad (3)$$

The algebra (3) for correlated bosons is a "q-boson algebra" [8, 9]. Indeed for zero coupling $\gamma = 0$ ($q = 1$) the *representation* which is (2) of the algebra (3) becomes a representation of the Heisenberg algebra for ordinary canonical bosons. Since $\sum N_j$ commutes with our proposal $\mathbf{H}_{\mathrm{kin}}$ for bosons, we can consider [8] $\mathbf{H} = \mathbf{H}_{\mathrm{kin}} - (\mu - 1)\sum N_j$ where μ is the chemical potential. Then with $d = 1$, the integrable form of $\mathbf{H}$ we can go on to consider in this Letter is

$$\mathbf{H} = -\frac{1}{2}\sum_{j=1}^{M}\{B_{j+1}^{\dagger} B_j + B_{j+1} B_j^{\dagger} - 2N_j\} - \mu\sum_{j=1}^{M} N_j \qquad (4)$$

under periodic boundary conditions $j + M = j$. But this Hamiltonian $\mathbf{H}$ is exactly the q-boson hopping model recently introduced and solved in Ref. [8].

The model (4) was solved by reference to a more fundamental "q-boson lattice model" considered in Ref. [9] and solved there by the quantum inverse method (or algebraic BA). But we now have a direct solution by algebraic BA [10], and the model appears to fill a natural place within the general theory of the $d \equiv 1$ integrable

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systems [9, 11, 12]. The continuum limit of the Hamiltonian (4) is [8] the Bose gas, solved at zero temperature $T = 0$ [13] and at $T > 0$ [14] exactly. The Bose gas has the well-known pairwise repulsive δ-function interaction of strength c, and c is the coupling constant. Attempts to introduce repulsive δ-function interactions on a lattice site lead to problems from the many particles possible on these lattice sites; but these problems are now avoided in (4) through the *correlation terms* described by the $\mathbf{P}_j$. Moreover, the collapse of the many bosons possible on a single site does not occur because the quantum integrability means that *fermion*-like quasiparticles, obeying Fermi statistics, are the collective excitations induced in the momentum space. Furthermore the exact solvability means that methods of conformal field theory [12, 15, 16] can be used to calculate *exactly* the critical exponent θ which determines the long-distance asymptotics of the correlation functions. We shall report in this Letter in some completeness these asymptotic expressions for the correlation functions of the q-boson hopping model (4) and shall relate these to the correlation functions of the more familiar XY model.

In the continuum limit in which $j\delta = x$ with $\delta \to 0$, if $B_j = \sqrt{\delta}\, b(x)$, $N_j = \delta N(x)$, one finds for all finite $\gamma \geq 0$ that $[b(x), b^\dagger(y)] = \delta(x - y)$ and $N(x) = b^\dagger(x) b(x)$ and the corresponding limit of (4) is the Bose gas with $c = 2\gamma$. By expanding the q-boson model (4) itself in powers of γ, we find that, for $\gamma N_j \ll 1$,

$$\mathbf{H} \to -\frac{1}{2}\sum_{j=1}^{M}\{b_{j+1}^\dagger b_j + b_{j+1} b_j^\dagger - \tfrac{1}{2}\gamma[b_{j+1} b_j^\dagger N_j + N_{j+1} b_{j+1} b_j^\dagger + b_{j+1}^\dagger (N_{j+1} + N_j) b_j] - 2N_j + O(\gamma^2)\} - \mu \sum_{j=1}^{M} N_j . \tag{5}$$

At $\gamma = 0$, the free boson limit, (4) becomes the *linear* hopping model of ordinary bosons. Moreover (5) shows that all orders of nonlinearity in terms of ordinary bosons are contained in (4), that γ is indeed the effective coupling constant for this strongly coupled system of ordinary bosons, and that, through the presence of the N_j, these bosons are strongly correlated.

The $\gamma \to 0$ limit of (4) is a lattice form of the free boson limit, $c \to 0$, of the Bose gas. On the other hand $c \to \infty$ and $\gamma \to \infty$ are distinct phases of interacting boson systems: $c \to \infty$ is the "impenetrable" Bose gas [13], but for $\gamma = \infty$ (4) is evidently $\mathbf{H} = -\frac{1}{2}\sum_{j=1}^{M}\{\phi_{j+1}^\dagger \phi_j + \phi_{j+1}\phi_j^\dagger - 2N_j\} - \mu\sum_{j=1}^{M} N_j$ expressed in terms of the "ladder" operators ϕ_j, $\phi_j^\dagger$, and N_j: $[N_j, \phi_i] = -\phi_j \delta_{ij}$, and H.c., but $[\phi_j, \phi_i^\dagger] = \pi_j \delta_{ij}$ the vacuum projector, $\pi_j = |0\rangle_j\langle 0|_j$.

We shall now establish the connection between the q-boson model (4) and the XY model of spin S. In a magnetic field h, the XY model has quantum Hamiltonian

$$\mathbf{H}_{XY} = -\frac{1}{4S}\sum_{j=1}^{M}\{\mathbf{S}_{j+1}^{+}\mathbf{S}_j^{-} + \mathbf{S}_{j+1}^{-}\mathbf{S}_j^{+}\} - h\sum_{j=1}^{M}\mathbf{S}_j^{z}, \tag{6}$$

and the $\mathbf{S}_j^{\pm}, \mathbf{S}_j^{z}$ satisfy the su(2) Lie algebra. By Holstein-Primakov transformation $\mathbf{S}_j^{-} = (\mathbf{S}_j^{+})^\dagger = b_j^\dagger\sqrt{2S - N_j}$, $\mathbf{S}_j^{z} = S - N_j = S - b_j^\dagger b_j$, $\mathbf{H}_{XY}$ has the well-known $1/S$ expansion which, however, is *identical,* to order γ, with the expansion of $\mathbf{H}$ Eq. (5): $\gamma \equiv 1/2S$ and μ is identified as $\mu = 1 - h$. We can thus conclude that the large S limit of the XY model is the q-boson hopping model (4) in the limit of small γ, as given by Hamiltonian (5).

To determine the actual asymptotics of the correlation functions for the q-boson hopping model (5) we first need results for the thermodynamics of the model. We give some of these next. The N-particle energy E_N is found by direct application of the algebraic BA to be [10] $E_N = \sum_{j=1}^{N}\{2\sin^2(p_j/2) - \mu\}$ and agrees with Ref. 8. The p_j are the roots of the Bethe equations $e^{-ip_jM} = \prod_{k=1,k\neq j}^{N} e^{i\Phi(p_j - p_k)}$ with the two-body phase shifts $\Phi(p) = 2\tan^{-1}\{\coth(\gamma)\tan(p/2)\}$. When $\gamma = \infty$ these Bethe equations have exactly determinable, purely algebraic, solutions [8].

The thermodynamics of the hopping model (4) is handled in the usual way [14]. The ground state energy of the model at finite temperatures $T = \beta^{-1}$ is determined through the quasiparticle excitation energies $\epsilon(p)$ above that ground state energy satisfying the nonlinear integral equations $\epsilon(p) = 2\sin^2(p/2) - \mu - (2\pi\beta)^{-1}\int_{-\pi}^{\pi} K(p - t)\ln(1 + e^{-\beta\epsilon(t)})\,dt$, $2\pi\rho(p)(1 + e^{\beta\epsilon(p)}) = 1 + \int_{-\pi}^{\pi} K(p - t)\rho(t)\,dt$, where $0 \leq \mu \leq 1$, and $K(p) = \partial\Phi(p)/\partial p = (\sinh 2\gamma)/(\cosh 2\gamma - \cos p)$. The function $\rho(p)$ is a quasiparticle density, and the pressure $\mathcal{P}$ of the system is equal to $\mathcal{P} = (2\pi\beta)^{-1}\int_{-\pi}^{\pi}\ln(1 + e^{-\beta\epsilon(t)})\,dt$ while the density $\mathcal{D} = \int_{-\pi}^{\pi}\rho(p)\,dp = \partial\mathcal{P}/\partial\mu$. At zero temperature $T = 0$ these integral equations become linear:

$$\epsilon_0(p) = 2\sin^2(p/2) - \mu + (2\pi)^{-1}\int_{-\Lambda}^{\Lambda} K(p - t)\epsilon_0(t)\,dt, \tag{7}$$

$$2\pi\rho_0(p) = 1 + \int_{-\Lambda}^{\Lambda} K(p - t)\rho_0(t)\,dt, \tag{8}$$

and $\epsilon_0(\pm\Lambda) = 0$, $-\pi \leq \Lambda \leq \pi$. The density $\rho_0(p)$ is now connected with the observable momenta of the quasiparticles $k(p)$, for $k(p) = p + \int_{-\Lambda}^{\Lambda}\Phi(p,t)\rho_0(t)\,dt$, and $2\pi\rho_0(p) = \partial k(p)/\partial p$. The pressure $\mathcal{P}$ and the density $\mathcal{D}$ now become

$$\mathcal{P}_0 = -(2\pi)^{-1}\int_{-\Lambda}^{\Lambda}\epsilon_0(p)\,dp > 0\,,\qquad \mathcal{D}_0 = \int_{-\Lambda}^{\Lambda}\rho_0(p)\,dp = k_F/\pi\,, \tag{9}$$

where $k_F = k(\Lambda)$ is the Fermi momentum. The low temperature expansion of the pressure is then $\mathcal{P} = \mathcal{P}_0 - (\pi/6v_F)T^2 + O(T^4)$ where v_F is the Fermi velocity: $v_F = [\partial\epsilon_0(p)/\partial k(p)]_{p=\Lambda} = \epsilon_0'(\Lambda)/2\pi\rho_0(\Lambda)$.

To obtain the critical exponents which determine the long-distance asymptotics of the correlation functions within the scaling limit we use the predictions from conformal field theory (CFT) [12, 15, 16]. The integral equation (7) means that at $T = 0$ the dispersion curve is linear near the Fermi surface k_F: $\epsilon(k) = (k - k_F)v_F$ and this means the CFT is applicable. From our result for $\mathcal{P}$ as $T \to 0$ quoted below (9) and Ref. [16] we can conclude that the central charge $\mathbf{c} = 1$. In this case, the physical fields $\Psi_n(t)$ can be expressed in the scaling limit as sums of the conformal fields and their critical exponents θ in terms of pairs of conformal dimensions $\Delta^{\pm}$ of the conformal fields, and these exhibit a continuous dependence on the parameters of the model [12, 15]. We shall here follow the outline given in Ref. [12]. The generic asymptotic formula for the zero temperature unequal time correlation function of the physical lattice field $\Psi_n(t)$ is

$$\langle\Psi_n(t)\Psi_1(0)\rangle = \sum_Q \mathbf{C}(Q)\,e^{-2ink_Fd}(n - iv_Ft)^{-2\Delta^+}(n + iv_Ft)^{-2\Delta^-} \tag{10}$$

where the Q are sets of quantum numbers, $Q = (\Delta N, d, N^{\pm})$, all integers, labeling [12] a complete set of states and thereby labeling the conformal fields in the expansion of the physical ones, and the $\mathbf{C}(Q)$, here left undetermined, are the coefficients in that expansion. The θ determine the conformal dimensions $\Delta^{\pm}$ by $2\Delta^{\pm} = 2N^{\pm} + (2\theta)^{-1}(\Delta N \pm \theta d)^2$. The critical exponent θ is *defined* through $\theta = 2Z^2(\Lambda)$ where $Z(p)$ is the solution of the integral equation $Z(p) - (2\pi)^{-1}\int_{-\Lambda}^{\Lambda}K(p - t)Z(t)\,dt = 1$ and $K(p)$, for the q-boson hopping model, is the kernel in Eqs. (7) and (8). We also use the alternative expression $\theta = (d/d\Lambda)\int_{-\Lambda}^{\Lambda}Z(t)\,dt$ for θ. The $\Psi_n(t)$ ultimately dictate [12] the choice of the Q in (10). From (8) it follows immediately that $Z(p) = 2\pi\rho_0(p)$ and from $\theta = (d/d\Lambda)\int_{-\Lambda}^{\Lambda}Z(t)\,dt$ we get the expression

$$\theta = 2\pi\,\frac{\partial\mathcal{D}_0}{\partial\Lambda} \tag{11}$$

from which we evaluate all of the θ's reported in this Letter.

To obtain the leading terms of the temperature dependent correlation functions at very small temperatures $T > 0$ we use the standard mapping $(n - iv_Ft) \to (v_F/\pi T)\sinh\{\pi T(n + iv_Ft)/v_F\}$ in (10). At equal times, $t = 0$, for simplicity, (10) then becomes

$$\langle\Psi_n(t)\Psi_1(0)\rangle_T = \sum_Q \mathbf{C}(Q)e^{-2ink_Fd}(2\pi T/v_F)^{2(\Delta^+ + \Delta^-)}\,e^{-n/\xi} \tag{12}$$

and the correlation length ξ is given by $\xi^{-1} = (2\pi T/v_F)(\Delta^+ + \Delta^-)$.

For our present purposes the leading terms in the asymptotics of the zero temperature unequal time correlator $\langle B_n(t)B_1^{\dagger}(0)\rangle$ for the q-boson field (we are using Heisenberg representation) correspond to the smallest conformal dimensions $\Delta^{\pm}$ in (10) and these have [12] $\Delta N = 1$, $d = 0, \pm 1$, and $N^{\pm} = 0, 1$. Consequently,

$$\langle B_n(t)B_1^{\dagger}(0)\rangle \sim |w|^{-1/\theta}\{\mathbf{C}_1 + \mathbf{C}_2(w^{-2} + \bar{w}^{-2}) + \mathbf{C}_3|w|^{-\theta}\cos(2\pi n\mathcal{D}_0)(w\bar{w}^{-1} + \bar{w}w^{-1})\} \tag{13}$$

in which $w = n - iv_Ft$, and the $\mathbf{C}$'s are constants. Likewise from (12) at $T > 0$ and equal times, $\langle B_nB_1\rangle_T \sim \mathbf{C}_1'(2\pi T/v_F)^{1/\theta}e^{-\pi Tn/v_F\theta} + \cdots$ with the correlation length $\xi_f = v_F\theta/\pi T$.

The leading terms of the asymptotics of the unequal times density-density correlator at $T = 0$ are determined by $\Delta^{\pm}$ with [12] $\Delta N = 0$, $d = 0, \pm 1$, and $N^{\pm} = 0, 1$. Thus from (10)

$$\langle\langle N_n(t)N_1(0)\rangle\rangle \equiv \langle N_n(t)N_1(0)\rangle - \mathcal{D}_0^2 \sim \mathbf{B}_1(w^{-2} + \bar{w}^{-2}) + \mathbf{B}_2|w|^{-\theta}\cos(2\pi n\mathcal{D}_0)\,, \tag{14}$$

while from (12) at $T > 0$, $\langle\langle N_nN_1\rangle\rangle_T \sim \mathbf{B}_1'(2\pi T/v_F)^2e^{-2\pi Tn/v_F} + \cdots$ with $\xi_d = v_F/2\pi T$. These results can be confirmed by calculations based on an exact solution [17].

All of these results for central charge $\mathbf{c} = 1$ apply equally to continuum models (the Bose gas) and the lattice models [model (4)]. The transitions from algebraic decay of the correlation functions at $T = 0$ to exponential decay at $T > 0$ and the infinite correlation lengths as $T \to 0$ show the expected "superfluid phase transition" at $T = 0$ for these models. Results are then distinguished by the forms of the critical exponents θ as given by Eq. (11). For the q-boson hopping model (4) we can find the exact expressions for θ as $\gamma \to \infty$ and in the free boson limit $\gamma \to 0$. When $\gamma \to \infty$, the kernel $K(p)$, in Eqs. (7) and (8), $\to 1$, and so the density is $\mathcal{D}_0 = \Lambda/(\pi - \Lambda)$ and the Fermi velocity is $v_F = (1 + \mathcal{D}_0)^{-1}\sin[\pi\mathcal{D}_0/(1 + \mathcal{D}_0)]$. Thus from (11) $\theta = 2(1 + \mathcal{D}_0)^2$. On the other hand for the free fermion limit $c \to \infty$ of the Bose gas $\theta = 2$ [12], and $\gamma \to \infty$ is indeed a new phase of the interacting boson system. Moreover in the classical limit [8], when $\gamma \to \infty$, $\mathcal{D}_0 \to \infty$, $\theta \to \infty$, $\xi_f = 2/T$, and $\xi_d \to 0$.

In the free boson limit $\gamma \to 0$ the situation is substantially more complicated. The Bethe equations reduce to singular integral equations from which it is possible to extract $\mathcal{D}_0 = \Lambda^2/8\gamma$ and $v_F = 2\sqrt{2\gamma\mathcal{D}_0}$. These results agree with those found for the continuum Bose gas with $c \Leftrightarrow 2\gamma$ and $c \to 0$. The critical exponent θ for the q-boson model is then, from (11), $\theta = 2\pi\sqrt{\mathcal{D}_0/2\gamma}$ so that $\xi_f = 4\mathcal{D}_0/T$. Then both these results apply to the Bose gas with $c = 2\gamma$, and these are new results for the Bose gas (compare Ref. [18]).

For the XY model as $S \to \infty$ we may argue that its critical theory is the q-boson hopping model theory as $\gamma \to 0$ because of the identity to order γ of the expansions (5), and that from (6), with $\gamma \Leftrightarrow 1/2S$. With the correspondence $\mathcal{D}_0 \Leftrightarrow S(1-\sigma)$ where $0 \le \sigma \le 1$ is the scaled magnetization, obtained through the Holstein-Primakov transformation, we must then have the correlators (13) and (14), as well as those for $T > 0$, now all expressed in terms of the $\mathbf{S}_j^{\mp}, \mathbf{S}_j^z$, with

$$\theta_{XY} = 2\pi S\sqrt{1-\sigma}, \qquad \xi_f = 4S(1-\sigma)/T, \qquad (15)$$

and with corresponding results in zero field h ($\sigma = 0$). For $\sigma = 0$ the results are largely consistent with those of Refs. [19,20], although it is easy to see that the postulated extrapolation from $S \le 3$ in Ref. [20] that $\theta_{XY} = 4S$ for all S cannot be correct: the model (6) is integrable when $S = \frac{1}{2}$ and this corresponds to the free fermion $c \to \infty$ impenetrable Bose gas limit of the Bose gas for which $\theta = 2$ for arbitrary density [12] [i.e., for arbitrary magnetic field h in (5)].

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