Conformal Field Theory and Hyperbolic Geometry

P. Kleban[1,2,3,*] and I. Vassileva[2,4]

[1]School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540
[2]The Laboratory for Surface Science and Technology, University of Maine, Orono, Maine 04469
[3]Department of Physics and Astronomy, University of Maine, Orono, Maine 04469 [†]
[4]Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01003
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We examine the correspondence between the conformal field theory of boundary operators and two-dimensional hyperbolic geometry. Considering domain boundaries in critical systems and the invariance of the hyperbolic length allows a new interpretation of the basic equation of conformal covariance. The scale factors gain a physical interpretation. We exhibit a fully factored form for the three-point function. An infinite series of minimal models with limit point $c = -2$ is discovered. A correspondence between the anomalous dimension and the angle of certain hyperbolic figures emerges.

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In this Letter, we establish several connections between the conformal field theory of boundary operators and two-dimensional hyperbolic geometry. First, by consideration of domain boundaries in two-dimensional critical systems, and the invariance of the hyperbolic length, we motivate a reformulation of the basic equation of conformal covariance. The scale factors gain a new, physical interpretation. They operate to keep the distance from the end of the domain boundary to the boundary of the system fixed. We also point out that for any geometry conformally equivalent to the half plane, domain boundaries in two-dimensional critical systems follow hyperbolic geodesics. Their energy per unit hyperbolic length is finite. Motivated by these results, we next exhibit a completely factored form for the three-point correlation function (and the prefactor of the four- or higher-point function). Here, a connection between the anomalous dimension of a primary operator and the angle of a hyperbolic figure appears. Finally, we impose the condition that the Schwarz function defined by a four-point function of operators degenerate at level two correspond to a hyperbolic tiling, or *tessellation*. This leads to a new, doubly infinite discrete set of minimal models. The angle-dimension correspondence is again encountered.

These results are of interest for several reasons. The new interpretation of the covariance equation, which appears in the context of conformal field theory in the half plane, lends credence to the idea that the half-plane realization of this symmetry is in some sense more physical [1]. The fact that domain boundary at criticality follows a hyperbolic geodesic and its energy is proportional to its *hyperbolic* length is, from the view point of statistical mechanics, quit surprising. It suggests the existence of a geometric action principle at critcality. The connections between the anomalous dimensions of primary operators and the angles of hyperbolic figures may indicate some deeper correspondence. The mapping from a certain set of (mainly) hyperbolic tilings to a specific series of minimal models lends support to this possibility.

To begin, we establish a connection between conformal field theory and hyperbolic geometry in the language of the theory of phase transitions. However, it should be emphasized that our results are generally valid, and not dependent on this particular realization of the theory.

As demonstrated elsewhere [2], a domain boundary in the upper half plane is created by boundary operators $\phi(x)$ [1, 3–5] located at its end points on the real axis. These operators act to change the boundary condition along the edge of the system [1], the real axis. Boundary operators may also be defined by letting bulk operator in a system with a boundary approach the boundary, and making use of the bulk-boundary operator product expansion [2, 4].

Although a domain boundary at a critical point exhibits large fluctuations, and has energy that is not proportional to its length, it is a well-defined object. Conformal invariance implies universality, which allows one to study it in generality. The (extra free) energy of such a boundary is

$$F = -\ln\langle\phi(x_1)\phi(x_2)\rangle, \tag{1}$$

as described in [2]. For completeness, we note that Eq. (1) ignores both universal [1, 6] and nonuniversal constants independent of x_1, x_2. The former are associated with the boundary states on the real axis, while the latter arise in computing the free energy of the boundary of any real system or statistical mechanical model.

Evaluating the correlation function, we find [2]

$$F = 2\Delta \ln|x_1 - x_2|, \tag{2}$$

where Δ is the critical dimension of ϕ. Now, Eq. (1) also gives the domain boundary free energy in any geometry conformally equivalent to the half plane, if we evaluate the correlation function in the new geometry. This is done by making use of the basic equation of conformal covariance of correlation functions [7], as applied to boundary operators,

$$\langle\phi_1(x_1)\,\phi_2(x_2)\cdots\rangle = |w'(x_1)|^{\Delta_1}\,|w'(x_2)|^{\Delta_2}\langle\phi_1(w_1)\,\phi_2(w_2)\cdots\rangle. \tag{3}$$

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Here $w = w(z)$ is an arbitrary conformal transformation, with $w_i = w(x_i)$.

Now, the domain boundary itself, in the half plane, is a half circle between the points x_1 and x_2. This fact follows by considering a single change of boundary conditions, at the origin, say. By symmetry, the corresponding boundary lies along the y axis. A projective transformation brings the end points to x_1 and x_2, and the straight line becomes a half circle. Such a half circle is precisely a geodesic of the hyperbolic, or Poincaré, metric $ds_h^2 = y^{-2}\, ds^2$ (for $y > 0$) [8]. The tendency of the domain boundary to avoid the real axis corresponds to the divergence of the hyperbolic length as $y \to 0$.

Next consider the hyperbolic geodesic between the points $z_1 = x_1 + \varepsilon_1$ and $z_2 = x_2 + \varepsilon_2$. Its hyperbolic length follows from standard results [8]

$$l_h = 2\ln|x_1 - x_2| - \ln\varepsilon_1\varepsilon_2, \qquad (4)$$

and diverges as z_1 or z_2 approaches the real axis. On the other hand, it is natural to define the domain boundary to begin and end at a finite but small (Euclidean) distance ε above the real axis, where ε will be on the order of the lattice spacing in any physical model. Then, up to additive constants, the boundary energy is proportional to the hyperbolic length,

$$F = \Delta(l_h + \ln\varepsilon^2). \qquad (5)$$

Note that factors giving rise to the $\ln\varepsilon^2$ term in Eq. (5) will appear naturally in any lattice calculation of F in a spin model, through the normalization of the conformal operators [9]. The fact that domain boundaries follow a hyperbolic geodesic and Eq. (5) suggest the existence of a geometric action principle for these quantities [10].

Next consider any other geometry that can be mapped to the half plane by a conformal transformation, for instance an infinite strip (with edges) of width L, $w = (L/\pi)\ln z$. Under any such transformation, the hyperbolic length is invariant. The hyperbolic metric in the new geometry is induced by the transformation. In the strip, for instance, $g = (\pi/L)^2/\sin^2(\pi v/L)$, where $w = u + iv$. Although the hyperbolic length of the boundary remains fixed, the transformation changes the distance between each end point and the edge of the system, by an amount proportional to the scale factor $|w'(x)|$. For the transformed theory to represent a physical domain boundary in the new geometry, one must readjust its end points to be at distance ε from the edges. Using the invariance mentioned and Eq. (1) then leads directly to Eq. (3), which is the basis of conformal field theory. The scale factors appear in (to our knowledge) an entirely new interpretation—they are necessary to insure that, in the new geometry, the boundary begins and ends in the appropriate place.

The argument of the preceding paragraph is not quite complete. The results described provide a hyperbolic interpretation for an arbitrary two-point correlation function of boundary operators. Specification of the full theory involves higher-point correlation functions. The new element that appears is their dependence on cross ratios [11]:

$$C = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_4)(x_1 - x_3)}. \qquad (6)$$

However, if we consider the points $z_i = x_i + i\varepsilon$, it is easy to see that

$$C = \exp\left[\tfrac{1}{2}\{l_h(1,2) + l_h(3,4) - l_h(2,4) - l_h(1,3)\}\right]. \qquad (7)$$

Now, as mentioned, l_h is invariant under conformal transformations. The combination that appears in Eq. (7) is, in addition, unaffected by the scale factors. Equation (3) thus follows immediately. It should be recognized that when more than one operator is included in the correlation function, a weighted sum of hyperbolic lengths will appear in the logarithm of the correlation function, with the weighting depending on the dimensions of the operators.

Note that in many cases four- (and higher-) point correlation functions can be interpreted in terms of domain boundary energies, including interactions [2]. In the case of critical percolation, an alternative interpretation of boundary operator correlation functions in terms of the probabilities of events is possible. This view has been exploited for the description of crossing probabilities in finite geometries [12–14].

The boundary operator theory is in general completely equivalent to the corresponding bulk conformal theory [4,5,15], in the sense that, for a given central charge, the spectrum of primary operators is the same. However, a given operator will generally play a different role than in the bulk.

It should be emphasized that the systems to which the theory applies are defined in flat space. The hyperbolic geometry, which describes a space of constant negative curvature, arises naturally from the mathematics, without having been put in at the start, by contrast to other treatments of field theories [16] and statistical systems [17] defined on hyperbolic spaces. From a mathematical point of view, this is not completely unexpected. To represent a physical theory, the metric must be equivalent at equivalent points in the half plane. Only two metrics satisfy this condition—Euclidean and hyperbolic. Of course this only means that the hyperbolic metric *can* occur, not that it necessarily will.

We pause to consider some implications of our results thus far. We have shown that the invariance of the hyperbolic length and natural physical requirements lead to a new derivation of the basic equation of conformal covariance, including a new interpretation of the scale factors. These considerations suggest that there is a fundamental mathematical connection between conformal field theory in the half plane and hyperbolic geometry.

Also, we have demonstrated that the energy of a domain boundary in any geometry conformally equivalent to the half plane [e.g., Eq. (5) of [2]] appears as a line integral along a hyperbolic geodesic. (In fact, it was precisely the search for a representation of the energy as a line integral that led to these results.) From a physical viewpoint, this is very surprising. At the critical point the boundary is strongly fluctuating—for instance, in a strip of width L, the boundary's width will be of the same order. There is absolutely no reason to expect that one of its intrinsic properties can be described as a line integral. A general picture of domain boundaries as independent, weakly interacting objects was established in [2]. The fact that the energy of a single boundary is proportional to its hyperbolic length, as described above, illuminates its nature further. More specifically, it indicates that a boundary, despite its large fluctuations, is in some sense additive.

Next consider an arbitrary three-point function. This has the form [7]

$$\langle \phi_l(x_1)\phi_m(x_2)\phi_n(x_3)\rangle = C_{lmn}\frac{1}{x_{21}^{\Delta_l+\Delta_m-\Delta_n}x_{32}^{\Delta_m+\Delta_n-\Delta_l}x_{31}^{\Delta_l+\Delta_n-\Delta_m}}, \quad (8)$$

where C_{lmn} is an operator product expansion coefficient, $x_{ji} = x_j - x_i$, and we have taken $x_1 < x_2 < x_3$. Now consider the hyperbolic triangle defined by the three points $z_i = x_i + i\varepsilon$, $i = 1,2,3$. Using the cosine law for hyperbolic triangles [8] it is then straightforward to show that the angles at points x_1, x_2, x_3 are of the form $a\varepsilon$, $b\varepsilon$, and $c\varepsilon$, respectively, with $a = 2x_{32}/x_{21}x_{31}$, etc. It follows that

$$\langle \phi_l(x_1)\phi_m(x_2)\phi_n(x_3)\rangle = C_{lmn}\left(\frac{a}{2}\right)^{\Delta_l}\left(\frac{b}{2}\right)^{\Delta_m}\left(\frac{c}{2}\right)^{\Delta_n}. \quad (9)$$

Note the association of anomalous dimension and angle in Eq. (9), and the fact that it is completely factored —each angle is raised to the power of the corresponding operator only, in contrast to Eq. (8). Transforming Eq. (9) to a new geometry, as above, reproduces the correct conformal covariance of the three-point function [Eq. (3)] if one readjusts each vertex of the transformed triangle to be at distance ε from the edge of the system. Equation (9) is also valid for conformally invariant systems in higher-dimensional half spaces [9].

If we let ϕ_n be the unit operator, Eq. (9) describes the two-point function, by a triangle with a fictitious point x_3. The resulting expression is not independent of x_3 unless $\Delta_l = \Delta_m$, thus establishing orthogonality.

One can express the prefactor of an arbitrary N-point function G as a product of N similar factors, by considering the hyperbolic N lateral defined by the N points taken at distance ε above the real axis, as above. The prefactor then constitutes a solution of Eq. (A.9) in [11]. The remaining factor in G is a function Φ of the N $-$ 3 independent cross ratios C_i. These quantities may also be expressed through hyperbolic angles. To see this, consider a four-point function. If, for instance, one draws the triangle defined by points x_1, x_2, and x_3, C is given by the ratio of the angle at x_3 of this triangle to the angle at x_3 of the quadrilateral.

Now consider a four-point function G of operators ϕ degenerate at level two, i.e., $\phi(1,2)$ or $\phi(2,1)$. This condition implies that the dimension Δ of ϕ is an algebraic function of the central charge c, and that the factor Φ is proportional to a hypergeometric function, i.e., there is a factor in G that satisfies a hypergeometric equation [11]. Now the ratio of two independent solutions of a hypergeometric equation defines the Schwarz function, which maps the upper half plane onto a triangle with curvilinear sides [18, 19]. In the present case, the triangle is equiangular, with angle $\pi|\Delta'|$, where Δ' is the dimension of the operator ϕ' (i.e., $\phi_{(1,3)}$ or $\phi_{(3,1)}$) appearing in the operator product expansion of ϕ with itself. Now one may reflect the triangle across any of its sides, which corresponds to a reflection of C across the real axis. Repeating this procedure gives rise to a set of curvilinear triangles that may overlap, i.e., the inverse map is not necessarily single valued. If we require single valuedness, the triangles will tile (or, in the hyperbolic cases—see below, *tessellate*) a circular region [19]. This condition (for either choice of ϕ) specifies $\Delta' = |l|$, $l = \pm 1, \pm 2, \pm 3, \ldots$. Since Δ' determines Δ, which in turn fixes the central charge, one arrives at the doubly infinite discrete series

$$c = 1 - 3\frac{(l-1)^2}{l(l+1)}. \quad (10)$$

Equation (10) specifies a set of minimal models, including the Gaussian model ($c = 1$, $l = 1$), the Ising model ($c = 1/2$, $l = 2$), critical percolation and dilute polymers ($c = 0$, $l = 3$) [20], dense polymers [21] and matrix models [22–24] ($c = -2$, $|l| = \infty$), and the Yang-Lee edge singularity ($c = -22/5$, $l = -5$) [25]. For $|l| > 3$, the sum of the angles is less than π, so the triangles are hyperbolic, and a tessellation is produced. The inverse map is an automorphic function of the corresponding triangular group. For $|l| = 1$, the triangle reduces to a great circle, and the group consists of one element. Similarly, $|l| = 2$ gives the (finite) dihedral group $\langle 2,2,2\rangle$ [19]. Both these cases correspond to spherical geometry, with total angle greater than π, while $|l| = 3$ gives rise to a triangular lattice in flat space (total angle π). The inverse map is an automorphic function of the group so defined in each case.

In summary, we have established several connections between the conformal field theory of boundary operators and two-dimensional hyperbolic geometry. A new interpretation of the basic equation of conformal covariance arises, we find a fully factored form for the three-point function, and a doubly infinite discrete series of minimal models with limit $c = -2$ is discovered. A correspondence between the anomalous dimension and the angle of certain hyperbolic figures emerges.

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*Electronic address: kleban@maine.maine.edu
†Present and permanent address.

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