Scenario for a Quantum Phase Slip in Superfluid ^{4}He

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(Received 20 July 1993)

Slips of the superfluid wave-function phase occur when the flow velocity through an aperture exceeds a critical threshold v_c. We propose a phase slip mechanism in which a microscopic vortex half ring is nucleated at the wall of the aperture. Its subsequent trajectory is described. The properties of the vortex close to the wall are inferred from the Gross-Pitaevskii equation and yield, at $T \sim 0.1$ K, a vortex radius at nucleation of 14 Å and a critical velocity of 21 m/s, in agreement with experiment.

PACS numbers: 67.40.Hf, 67.40.Vs

At absolute zero, superfluid ^{4}He behaves as an inviscid one-component fluid which can sustain vortices quantized in units of the quantum of circulation κ_4. The existence of these vortices was postulated by Onsager, and independently by Feynman [1], who also suggested that they could account for the phenomenon of critical velocity in superfluid flows. These ideas were carried out further by Anderson [2], who introduced the concept of slippage of the phase of the quantum wave function: vortices are viewed as singularities in the flow field about which the quantum phase turns by 2π and which move about, exchanging energy with the potential flow as they grow or shrink. Direct evidence for these properties is provided by phase slippage experiments [3–5] in which discrete dissipation events in the flow of ^{4}He through a microscopic aperture are interpreted as slips of the quantum phase difference across the aperture by 2π.

More recent experiments [6–8] have confirmed the early findings and have revealed a strong temperature dependence of the velocity threshold for phase slips, v_c, going approximately as $(1 - T/T_0)$ with $T_0 \sim 2.45$ K, and reaching a plateau below ~ 0.15 K [8,9]. This temperature dependence has been interpreted as resulting from thermal and quantum nucleation of nanometer size vortices on the walls of the aperture. This interpretation is further borne out by a detailed study of the vortex nucleation rate which stems from the statistical properties of v_c [8,10]. A last but essential piece of information has been provided recently by the study of the effect of minute concentrations of ^{3}He impurities on v_c [11]. It is estimated that, in these experiments, the local superflow velocity u_s at the vortex nucleation site reaches values of the order of 22 m/s at low temperature. In this Letter, we first address the problem of the properties of quantum vortices in the proximity of walls with the aim of accounting for such a value of the velocity and for its temperature dependence. We then describe a plausible scenario for 2π phase slips in which the vortices are nucleated very close to the walls and propagate across the aperture along well defined trajectories leading to reproducible discrete dissipation events.

A simple model of quantized vortices is provided by the Gross-Pitaevskii (GP) equation [12]. This equation, although it offers only a coarse description of superfluid ^{4}He, has been used in many discussions of quantized vortex formation and motion (see, e.g., [13–16]). We need further knowledge on the properties of vortices close to walls, which we obtain by considering a straight vortex filament standing perpendicularly on a flat inert boundary. This problem possesses axial symmetry and we look for a solution to the GP equation of the form $\Psi_0 = e^{i\varphi} f(r,z)$ with $\nabla\varphi = 1/r$. The amplitude of the wave function f satisfies the following equation:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} + \left(1 - \frac{1}{r^2}\right) f - f^3 = 0. \qquad (1)$$

The same reduced units as GP are used, namely, r and z are in units of the GP core parameter a_0 and f^2 is in units of ρ_s/m, the bulk superfluid number density, i.e., f tends to 1 far from the wall and the vortex core. The part $\mathcal{F}$ of the thermodynamic potential density which depends on the superfluid order parameter reads (in units of the condensation energy density $\frac{1}{2}\rho_s c^{\star 2}$, the GP "sound" velocity being $c^\star = \kappa_4/2\pi\sqrt{2}a_0$ [17]):

$$\mathcal{F} = \left\{ f^4 - 2f^2 \left[1 - (\nabla\varphi)^2\right] + 2(\nabla f)^2 \right\}. \qquad (2)$$

With no vortex present, the wave-function amplitude f heals as $\tanh(z/\sqrt{2})$. In the presence of the vortex, we have solved Eq. (1) by a Gauss-Seidel iteration scheme under the condition that f be zero at the wall ($z = 0$) and on the vortex core ($r = 0$), and 1 far away. If we compute the difference in the thermodynamic potential densities integrated over r in a layer δz at a height z with and without vortex, we find the line energy $\delta E_0/\delta z$ of the vortex at height z. This quantity varies, for large r, as $\tanh^2(z/\sqrt{2}) \ln r/a_0 + L_0$. The quantity L_0, plotted in Fig. 1, represents the "core" energy and tends to the

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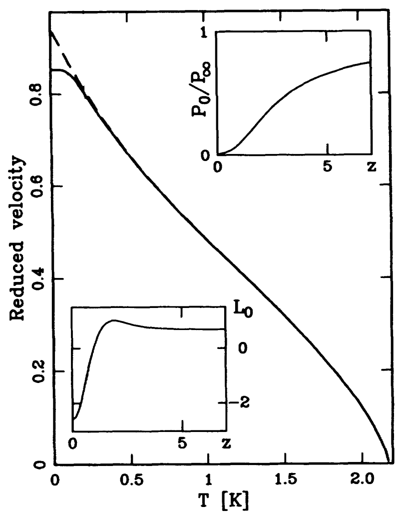

FIG. 1. Critical velocity in units of c^* vs temperature, calculated for the wall nucleation of a half-ring vortex whose energy L_0 per unit length, in units of $\rho_s\kappa_4^2/4\pi$, and impulse, normalized to $P_\infty = \pi\rho_s\kappa_4(za_0)^2/2$, are computed from the Gross-Pitaevskii equation and are shown in the insets vs z, the distance from the wall in units of a_0. The integration over r which yields L_0 is cut off at $r = 20$. The plain curve includes the effect of quantum tunneling [8].

GP value (0.38) away from the wall. At small z, it becomes negative and large. As f behaves asymptotically at large r as $(1-r^{-2})^{\frac{1}{2}}\tanh\{[\frac{1}{2}(1-r^{-2})]^{\frac{1}{2}}z\}$, L_0 diverges logarithmically with r as $z \to 0$.

This negative line energy arises for the following reason. From Eq. (2), it is seen that, close to the wall where $f \ll 1$, the potential density is dominated by $(\nabla f)^2$. Close to the foot of the vortex on the wall, both the wave-function amplitude and its gradient are depressed with respect to the hyperbolic tangent behavior. Hence the energy is reduced with respect to the vortex-free case, resulting in a negative line energy for the vortex: a vortex line very close to the wall costs no energy to form. Hence, we are led to conjecture the existence, even for small or vanishing local flow fields u_s, of a vorticity layer at the wall extending over a thickness of $\sim 3a_0$, as can be seen by integrating $\delta E_0/\delta z$.

Whether this vorticity layer manifests itself otherwise than on the nucleation of vortices is unclear. Core parameters of vortices in films have been found to be $\sim$ 10–20 Å [18], values to be contrasted with those in the bulk, $\sim$ 1 Å [19]. Surface tension measurements lead to $a_0 \simeq 5$ Å to 9 Å at low temperature [20], a value larger than the superfluid healing length $\sim$ 2–3 Å [21]. Wall phenomena in superfluid ^{4}He do not seem to be well described in terms of a single healing length. In the solution to the GP equation discussed above, the "effective" core radius increases as $z \to 0$.

The vortex nucleation process itself, i.e., the escape of vortices from this layer at some well defined location on the wall, the nucleation site, is now treated in the same fashion as in Ref. [8] with the following important modification: the vortex properties close to the wall are now *derived* from the above solution to the GP equation instead of being assumed classical with an *ad hoc* cutoff close to the wall. We assume as in our earlier work [8] and following a number of authors [15,22,23], that the nucleated vortex appears over a flat or nearly flat wall in the approximate shape of a half ring extending perpendicularly from the wall. When placed in a local velocity field u_s, the vortex "free" energy is given in terms of its energy E_0 and its impulse P_0 at rest by

$$E_v = E_0 - P_0 u_s, \tag{3}$$

assuming that the vortex self-velocity opposes the flow and effectively decreases the energy. The impulse of the half ring is expressed by [15,24]

$$P_0 = -\int dS \rho_s \kappa_4 f^2. \tag{4}$$

The integral is taken over the flat surface spanned by the half ring and limited by the wall. Its outcome, as a function of z $(=R)$, is shown in Fig. 1.

The half-ring energy E_0 is half that of the full ring which, in the bulk, is expressed by $\frac{1}{4}\rho_s\kappa_4^2 R \times \{\log(R/a_0) + L_0\}$, the core energy L_0 being very nearly equal to 0.38, namely, the same value as for straight filaments in the bulk. This near equality also holds for a half ring of large radius as the wall affects only two short and nearly straight portions close to its feet. In the following, we shall neglect for small radii the self-influence of the ring on its core energy which we shall write as π times the integral of L_0 from 0 to $R = z$ [25].

The vortex free energy E_v can then be computed for various values of u_s. The energy barrier is taken as the difference between the maximum value of E_v and the top of the energy band of the vortex boundary layer (namely, the zero for energies). The value of the applied flow field velocity necessary to overcome the energy barrier in the presence of thermal fluctuations at temperature T is computed as in Refs. [5] and [8]. The outcome of the calculation with an effective core radius of 4.7 Å is given in Fig. 1. At the quantum crossover temperature $T_q = 0.15$ K, the nucleated vortex has a radius of $\sim$ 14 Å and a self-velocity of 21 m/sec. This result reproduces both the T dependence observed experimentally [4,6–11] and the measured value of the local critical velocity u_c [11] with the same value of effective core radius. Furthermore, this value turns out to be close to the vortex layer thickness estimated above and which fixes the length scale relevant to vortices close to walls.

Once nucleated, the vortex survives in the applied flow field and flows away with the fluid particles, as required by the Kelvin-Helmholtz theorem. It can disappear only by a fluctuation process inverse of that of its nucleation.

Vortex trajectories by orifices have been the object of many studies [26] but it is only very recently that the 3D fluid-dynamical problem of half-ring vortices in the proximity of an aperture has been addressed [27]. We shall here gain some insight into the typical trajectories in such a situation with the help of two analytical examples.

First, we consider the case of a half-ring vortex nucleated at $T \simeq 0$ (i.e., we neglect normal fluid) in the close vicinity of a pointlike aperture in an infinite plane, the axis of the half ring going through the aperture. The streamlines of the applied flow are straight lines fanning out from the aperture. The corresponding velocity varies as the inverse of the square of the distance ζ from the source, $J/2\pi\rho_s\zeta^2$, J being the mass flux. Each point of the half-ring vortex, whose self-velocity varies as the inverse of its radius R (to the neglect of small logarithmic terms), moves in a fixed plane containing the axis of symmetry [28]. At first after nucleation, the vortex is pushed along the flow away from the nucleation site, but since its self-velocity decreases only as R^{-1} while that of the potential flow varies as ζ^{-2}, the direction of motion is reversed after some distance: as pictured in Fig. 2, the half ring comes back toward the aperture, flies over it, and away from it with a steady motion towards infinity. Its final size is such that the total line energy is exactly $\Delta E = \kappa_4 J$, as required by the cut-flux theorem [29].

If the size d of the aperture is finite instead of being infinitely small, trajectories such as the ones depicted in Fig. 2 are possible only under the condition that the radius of the half ring be greater than $d/2$ when it caps over the aperture (i.e., at the point at which it has cut half of the streamlines and where its energy is $\kappa_4 J/2$). When expressed in terms of the critical velocity through the aperture, $v_c = 4J/\pi\rho_s d^2$, such a condition reads (with $d \gg a_0$)

$$v_c \geq \frac{\kappa_4}{\pi d}\left[\ln\left(\frac{8d}{a_0}\right) - \frac{1}{4}\right]. \qquad (5)$$

Equation (5) expresses Feynman's criterion [1,5] for the critical velocity through orifices of size d and appears here as a geometrical constraint on the half-ring final size.

To illustrate further the possible trajectories of a vortex over an aperture, we consider, as our second case example, a straight vortex filament moving over an infinitely long slit, the filament axis remaining parallel to the slit. This simple two-dimensional (2D) situation can be solved by conformal mapping [30] and the results are shown in Fig. 3. Two cases (and two only) are seen to occur, according to the initial conditions: (1) either the flux is large enough to push the vortex away (from the wall and its image), across the slit and to the left of the figure, (2) or it is not, and the filament turns about the slit edge and creeps along the plate to the right of the figure.

This typical asymptotic behavior of the vortex trajectory also holds in more general 3D flows: the vortex gains energy when its self-velocity combines with that of a divergent flow in such a way that it moves away from its image (in 2D and 3D) and grows (in 3D); in the opposite case, it loses energy. In the latter situation, it shrinks back to a very small size, or very close to its image, and simply amounts to a large scale fluctuation in the flow. In the former situation, it ends up a large distance from the aperture having cut all the streamlines of the applied flow field and having collected an energy $\Delta E = \kappa_4 J$, in accordance with the ac Josephson relation [3]. Ultimately, this vortex will dissipate its energy by radiating phonons when its flow field suffers distortion on obstacles [16]. Thus, the scenario for a 2π phase slip that we propose here consists, as a first step, of the thermal or quantum escape of a half ring out of the vorticity layer

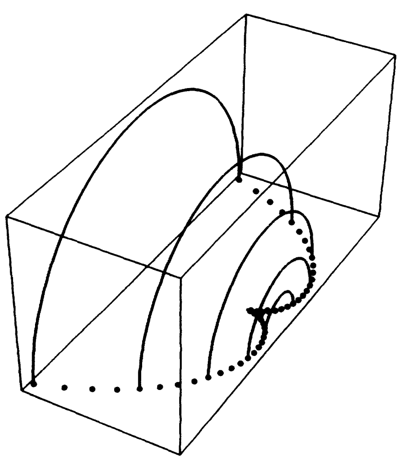

FIG. 2. Three-dimensional plot of the half-ring vortex at various times after nucleation at the point source. The vortex leaves the picture to the back in the upper left direction and moves to far-away distances while changing little in radius.

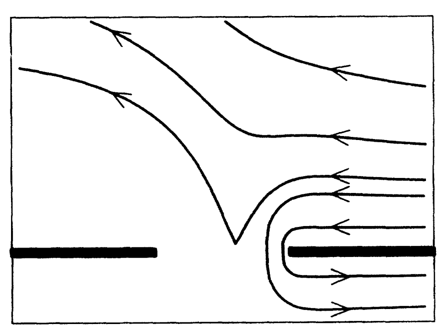

FIG. 3. Vortex trajectories over a two-dimensional slit, represented by the heavy lines. The vortex is assumed to be created at some initial distance from the wall, above it and to the right of the figure. Arrows mark the direction of evolution. The trajectory with a cusp separates the two types of vortex behavior discussed in the text. For this picture, the mean (upward) applied flow velocity is $0.2\kappa_4/d$, d being the slit width.

at a nucleation site on the wall, its subsequent trajectory then being such that it caps over the aperture, cuts all the potential flow lines, and carries away a reproducible lump of energy.

We wish to express our thanks to D. D'Humières, Y. Pomeau, E. Sonin, G. Volovik, and G. Williams for fruitful discussions and correspondence, and more particularly to K. Schwarz for having undertaken the three-dimensional vortex motion simulation in short orifices at our request [27].

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