Fermions in Quantum Gravity

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(Received 23 February 1994)

The quantum fermions plus gravity system (QGD) is studied using the loop representation. A Hamiltonian is constructed that governs the evolution in the physical time given by a ("clock") scalar field. The Hamiltonian, defined via a regularization, is finite, background independent, and diffeomorphism invariant; it acts on intersections and end points of open curves. The dynamics is thus coded into combinatorics of graphs ("topological Feynman rules"). Exact dynamical states are exhibited. Surprisingly, the fermion dynamics is the immediate extension of pure gravity dynamics to open loops.

PACS numbers: 04.60.Ds

Recent results in nonperturbative quantum gravity—definition of general covariant regularization techniques [1] and construction of a finite physical-time evolution Hamiltonian [2]—have been achieved using the loop representation [3,4] in the context of *pure* gravity. An important question is whether these results and their geometrical simplicity survive in the presence of matter: not only because matter fields are constituents of a realistic theory, but also because the presence of matter drastically simplifies the key problem of defining physical observables in a general covariant theory [5]. Thus, an important problem in quantum gravity is to construct a generally covariant description of the dynamics of matter at the Planck scale [6]. In this Letter we show the following: (i) The inclusion of fermions in the loop representation can be achieved in a natural way (as one may have expected) by extending loop space to include open curves. (ii) The diffeomorphism invariant quantum states of the fermion-gravity system are classified by graphs containing open and closed lines. (iii) Physical-time evolution can be implemented as in [2] by further coupling a ("clock") scalar field to the system: the resulting Hamiltonian is diffeomorphism invariant, finite, and acts in a purely combinatorial fashion on the graphs—its action can be defined in terms of a simple set of "topological Feynman rules." A diffeomorphism invariant picture of fermionic matter and gravity at the Planck scale emerges from here. We describe the simplest dynamical states at the end of this Letter. In addition, we describe what we consider a strange and surprising aspect of the loop representation: (iv) The dynamics of the coupled gravity-fermion system can be obtained by just extending to open curves the action of the *pure gravity* Hamiltonian (defined for closed curves). We have not been able to find any convincing interpretation of this fact.

We consider the Einstein-Weyl theory for a massless, two-component fermion field coupled to gravity. In the Ashtekar formalism [7,8], this theory is defined by the constraints

$$G_{AB} = -i\sqrt{2}\mathcal{D}_a\sigma^a{}_{AB} + \pi_{(A}\psi_{B)}, \tag{1}$$

$$V_a = i\sqrt{2}\,\sigma^{bAB}F_{abBA} - \pi_A\mathcal{D}_a\psi^A, \tag{2}$$

$$\begin{aligned} C &= \sigma^{aAB}\sigma^b{}_B{}^C F_{abCA} + i\sqrt{2}\sigma^a{}_A{}^B\pi_B\mathcal{D}_a\psi^A \\ &= C_{\rm gr} + C_{\rm Weyl}, \end{aligned} \tag{3}$$

on the phase space coordinated by the Ashtekar variables A_{aAB} and $\sigma^a{}_{AB}$, and by the fermionic (Grassmann-valued) canonically conjugate fields ψ^A, π_A. Here $A, B = 1,2$ are spinor indices and $a, b = 1,2,3$ are space indices. See [4,9] for notation and details. The minimally coupled fermion interaction in the Ashtekar formalism is not equivalent to the minimally coupled fermion interaction in the metric variable. Rather, it corresponds to the Einstein-Cartan-Weyl theory [10]—see [11], and references therein.

We begin by introducing "loop" variables. These include the loop variables $T[\alpha], T^a[\alpha](s)$ of the purely gravitational case [3], and the "open loop" variables [12]

$$X[A,\alpha] := \psi^A(\alpha_i)\,U_{\alpha A}{}^B\,\psi_B(\alpha_f), \tag{4}$$

$$Y[A,\alpha] := \pi^A(\alpha_i)\,U_{\alpha A}{}^B\,\psi_B(\alpha_f)\,, \tag{5}$$

where α is a single *open* curve $\alpha : s \mapsto \alpha^a(s)$ (which we improperly insist on denoting as "loop"), with end points α_i and α_f, and $U_\alpha[A]$ is the parallel transport matrix. These variables form a closed Poisson algebra among themselves and with the T variables, as can be directly verified. They are reparametrization invariant and satisfy the same retracing and spinor identities as the T's [4]. The algebra can be expressed in terms of breaking and rejoining of loops at intersections and of gluing of loops at end points. As in the pure gravity case, we can also consider "higher order" variables, as

$$Y^a[A,\alpha](s) := \pi^A(\alpha_i)\,U_{\alpha A}{}^B(0,s)\,\sigma^a{}_B{}^C(\alpha(s))U_{\alpha C}{}^D(s,1)\,\psi_D(\alpha_f). \tag{6}$$

0031-9007/94/72(23)/3642(4)$06.00

The quantum theory is defined by a linear representation of the Poisson algebra of the loop observables [3,13]. This is given by operators acting on wave functionals $\Psi[\beta]$ depending on sets β of open and closed curves (multiloops). The open loops can be seen as the continuum limit of the Wilson-Kogut-Suskind flux-tube states in lattice QCD [14]. The quantum operators corresponding to the open loop variables are

$$\hat{X}[\alpha]\Psi[\beta] = \Psi[\alpha \cup \beta], \quad (7)$$

$$\hat{Y}[\alpha]\Psi[\beta] = i\sum_e \delta^3(\alpha_i, \beta_e)\Psi[\alpha^{-1} \cdot_e \beta], \quad (8)$$

$$\hat{Y}^a[\alpha](s)\ \Psi[\beta] = i\sum_e \delta^3(\alpha_i, \beta_e)\Delta^a[\alpha,\beta](s) \times \sum_{q=\pm} \Psi[\alpha^{-1} *_e *_s{}^q \beta]. \quad (9)$$

Here e labels end points β_e of the multiloop β. The notation $\alpha \cdot_e \beta$ indicates loops' composition (gluing at coincident end points), and $\alpha *_e *_s{}^q \beta$ indicates a double grasping [3] between α and β: one through β_e and the other at $\alpha(s)$, where q labels the two possible reroutings. See [4,11] for other details on notation.

The quantum version of the diffeomorphism constraint (2) is the generator of the natural action of the diffeomorphism group on the space of open and closed loops [3]. Its general solution is $\Psi[\alpha] = \Psi[K(\alpha)]$ where K is a *generalized* knot class, defined as a diffeomorphism equivalence class of sets of open and closed lines. The knot states, which have support on a single class, can be (over) characterized by the following: the number of end points N, intersections I, and their orders $m_1 \dots m_I$ (number of lines emerging from the intersection), the moduli-space parameters of the intersections $a_1^{m_1}, \dots, a_I^{m_I}$, and the "braiding" $\mathcal{K}_M$, where $M = \sum_i m_i$, obtained by erasing the intersection points [2]. They can be denoted as $|N, I, a_1^{m_1} \dots a_I^{m_I}; \mathcal{K}_M\rangle$. The representation is defined by the assumption that these knot states have finite norm [15]. The total fermion "charge" $N = \int \pi_A \psi^A$ is a diffeomorphism invariant conserved quantity. The corresponding quantum operator can be constructed as the space integral of the limit of $\hat{Y}[\alpha]$ when α shrinks to a point. The result is the fermion-number operator, and one can directly check that N—the number of end points—is its quantum number. This confirms the natural interpretation of the number of open ends as the number of fermions in the state.

Let us now extend the theory by (minimally) coupling a further scalar field $T(x)$, with the aim of using it for defining a physical internal time, as in [2,5]. This turns the Hamiltonian constraint into a genuine Hamiltonian. By fixing the gauge $\partial_a T(x) = 0$, and restricting to the clock regime $\partial_t T(x,t) > 0$, we obtain a genuine Schrödinger equation $i\hbar\partial_T \Psi(T) = \hat{H}\,\Psi(T)$ that governs the evolution of the gravity-fermions degrees of freedom in the constant $T(x) = T$ hypersurfaces. The Hamiltonian turns out to be $H = \int d^3x\, \sqrt{-C}$. We refer to [2] for the details of this construction. The problem is to find a finite and general covariant definition of the operator $\hat{H}$. To this aim, following [1], we introduce a fictitious background flat metric and a preferred set of coordinates in which this metric is Euclidean, and we write

$$H = \lim_{\substack{L\to 0\ \delta\to 0 \\ A\to 0\ \tau\to 0}} \sum_I L^3 \sqrt{-C_{\mathrm{gr}\,I}^{A,L,\delta} - C_{\mathrm{Weyl}\,I}^{L,\tau,\delta}}\,. \quad (10)$$

We have partitioned three-dimensional space into cubes of side L, labeled by the index I. The quantity $C_{\mathrm{gr}\,I}^{A,L,\delta}$ is the regularized form of the pure gravity Hamiltonian constraint C_{gr}, as defined in [2]. Next, we define the open loop $\gamma^\tau_{\mathbf{x},\mathbf{y}}$, where τ is a regularization parameter, $\mathbf{x}$ is a point in space, and $\mathbf{y}$ is a vector in the tangent space around $\mathbf{x}$, as the (uniformly parametrized) straight line (in the background metric) that starts at $\mathbf{x}$ in the $\mathbf{y}$ direction and is long τ. Using this, we define the regularized form of the fermion component of the Hamiltonian as

$$C_{\mathrm{Weyl}\,I}^{L,\tau,\delta} = \frac{1}{L^3}\int_I d^3x\; C_{\mathrm{Weyl}}^{\tau,\delta}(\mathbf{x})\,, \quad (11)$$

$$C_{\mathrm{Weyl}}^{\tau,\delta}(\mathbf{x}) = c_{\tau\delta}\int d^3y\; \theta(\delta - |y|)\; \frac{y^a}{|\mathbf{y}|}\; Y^a\left[\gamma^\tau_{\mathbf{xy}}\right](|y|/\tau), \quad (12)$$

where $c_{\tau\delta} = (3/\tau)(1/\frac{4}{3}\pi\delta^3)$ and $\theta(x)$ is the step function. Note the role of the regularization parameters: L fixes the size of the boxes. τ gives the length of the "small" loop. The direction of this loop is integrated over (d^3y angular integration). Y^a has a special point where the σ is inserted (the "hand"). The position of this point is also integrated over (d^3y radial integration). The point splitting regularization is implemented by this smearing of the position of the hand, and its size is determined by δ. By expanding in τ and δ one verifies that H so defined provides a genuine regularization of the Hamiltonian. We define the quantum Hamiltonian by replacing Y^a in (12) with the corresponding quantum operator (9). The computation of the action of the resulting operator on a loop state is again a straightforward exercise, yielding (we temporarily put $\hat{C}_{\mathrm{gr}} = 0$)

$$\hat{H}\,\Psi[\alpha] = \lim_{L,\delta,\tau\to 0} \sqrt{\frac{9\,L^3}{4\pi\,\delta^2\,\tau}}\, \sum_e \left(\hat{\mathcal{F}}_e^{\tau\delta}\right)^{-\frac{1}{2}}\, \Psi[\alpha], \quad (13)$$

$$\hat{\mathcal{F}}_e^{\tau\delta}\,\Psi[\alpha] = \sum_{l_e}\sum_{q=\pm} \Psi[\alpha * *^q{}_{e,\delta}\gamma^\tau_{\mathbf{x}(\mathbf{x}+\delta l)}], \quad (14)$$

where e labels end points of α and l_e are the tangents of the lines emerging from the eth end point—these lines can be more than one if the end point is not *free,* namely, if α is not injective at the end point.

The operator $\hat{H}$ is well defined only if it is finite and independent from the background metric used for the regularization. We observe that powers of lengths of the regularization parameters L, δ, τ in the prefactor in (13) cancel, a necessary condition for a finite limit. This cancellation is a nontrivial result that can be traced to the fact that H is diffeomorphism invariant, and that we are regulating "the square root of the square of a distribution," which, in a sense, is homogeneous of degree zero in the divergent factors. To complete the definition of $\hat{H}$ we have to fix the order in which the limits are taken (the choice amounts to a quantum ordering problem). For consistency with the above definitions we must have $\tau > \delta$, and, in order to avoid "boundary effects" in the box, $L > \delta$. We introduce a parameter ϵ, and put $L(\epsilon) = k\epsilon^3 a$, $\tau(\epsilon) = \epsilon\, a$, and $\delta(\epsilon) = \epsilon^4 a$, where a is an arbitrary length, and k is an arbitrary dimensionless positive number. We can now take the $\epsilon \to 0$ limit, yielding

$$\hat{H}\Psi[\alpha] = \lambda^2 \sum_{\alpha_e} \left(\hat{\mathcal{F}}_e\right)^{-\frac{1}{2}} \Psi[\alpha], \tag{15}$$

where we have introduced the "end-point operator"

$$\hat{\mathcal{F}}_e\Psi[\alpha] = \lim_{\epsilon\to 0} \hat{\mathcal{F}}_e^{\tau(\epsilon)\delta(\epsilon)}\Psi[\alpha], \tag{16}$$

and $\lambda = (\frac{3k^{-3/2}}{2\sqrt{\pi}})^{1/2}$ is a free constant that emerges from the regularization. Since δ goes to zero faster than τ, we can just take the $\delta \to 0$ limit first, and the $\tau \to 0$ limit second. Let us consider the $\delta \to 0$ limit of $\hat{\mathcal{F}}_e^{\tau\delta}\Psi[\alpha]$ (with finite τ). If the end point is free, the action of the operator is simply to add a small straight line of length $\tau = \epsilon a$ to the end point of the loop, in the direction of the incoming loop. If the end point is not free, the action of the operator produces one term for each component of α emerging from the end point. These terms imply the addition of the line and also a rerouting through the intersection, the pattern of which is given by (14). Before taking the limit $\tau \to 0$, let us assume that $\Psi[\alpha]$ is a diffeomorphism invariant state. If the end point α_e is free, we simply have $\lim_{\tau\to 0}\hat{\mathcal{F}}_e^{\tau 0}\Psi[\alpha] = 2\Psi[\alpha]$, because for small enough τ the added loop will not intersect any other loop, and the addition of a small line at the end of a loop does not change the knot class of the loop. If α_e is not a free end point, then $\alpha * *\gamma^{\tau}_{\alpha_e, l_e}$ *does* belong to a different knot class than α. But in any case, since $\Psi[\alpha]$ is diffeomorphism invariant, for small enough τ we have that $\hat{\mathcal{F}}_e^{\tau 0}\Psi[\alpha]$ becomes *independent from* τ. The limit is thus the limit of a constant function and therefore is *finite*. Moreover, it is clear that the resulting action of $\hat{\mathcal{F}}_e$ is well defined on the diffeomorphism invariant states. Thus, the operator $\hat{H}$ is finite and diffeomorphism invariant in the limit. If we now reinstate $\hat{C}_{\rm gr} \neq 0$, we have

$$\hat{H} = \sum_{i,e} \sqrt{\hat{M}_i + \lambda\hat{\mathcal{F}}_e}\,, \tag{17}$$

where i labels the intersections and $\hat{M}$ was constructed in [2]. $\hat{H}$ is a finite operator defined on knot states. We expect that the square root could be computed order by order as the complexity of the knots increases. This has been verified only in the simplest cases and work is in progress in this direction.

We are now in the position to describe the general structure of quantum gravitational dynamics, the quantum theory of gravitationally interacting fermions evolving in the clock time defined by a scalar field. A physical quantum state $|K\rangle$ of the theory is specified by a generalized knot (a graph with open ends). The quantum dynamics is given by the matrix $\hat{H}$ in knot space, given in Eq. (17). The matrix elements of the operators $\hat{M}_i$ and $\hat{F}_e$ can be directly computed between any two given knot states—from (14) and (16), and Ref. [2]. The calculation amounts to an exercise in the combinatorics of breaking and rejoining of loops at intersections. This action can be coded in a small number of simple graphic rules ("topological Feynman rules"), which we will publish elsewhere. These matrix elements determine the first order transition amplitudes in a time-dependent perturbation expansion in the clock time. In principle, exponentiation of the $\hat{H}$ action gives the full evolution.

For instance, we can start from the simplest state formed by a single non-self-intersecting open line. This can be denoted as $|2,0;1_2\rangle$, or, graphically, as $|\bullet\!\!-\!\!\bullet\rangle$. There are two fermions in this state. We have immediately $\hat{H}\;|2,0;1_2\rangle = 2\lambda|2,0;1_2\rangle$: this is a stationary state. Equivalently, the time-dependent Schrödinger quantum state,

$$|2,0;1_2;\;T\rangle = \exp^{i\lambda\sqrt{\frac{c^5}{\hbar G}}\;T}\;|2,0;1_2\rangle \tag{18}$$

(we have reinstated conventional units), is a solution of the *exact* quantum interacting theory. It is suggestive to think of this state as a simple fermion-gravity quantum configuration, with only two fermions gravitating around each other in the simplest of the quantum geometries, or a kind of "atomic" "ground state" (minimal energy in the clock time) of a simple two-fermion state. Other eigenstates are given by a set of n disconnected open lines $|2n,0;1_{2n}\rangle$. The corresponding energy eigenstates are $E_n = n\lambda\sqrt{c^5\hbar/G}$. As soon as we consider *intersecting* states, the complexity of the operators $\hat{M}_i$ and $\hat{F}_e$ becomes relevant, and we have nontrivial time evolution.

Finally, let us come to the relation between the full theory and pure gravity. A detailed investigation shows that the action of the Hamiltonian on end points (fermion term) can be seen as the extension of its action on intersections (gravity term). Rather than describing these similarities, we present a simpler formal account of the result, which disregards regularization issues. We recall from [3] that the action of the pure gravity Hamiltonian constraint on loop states is given by the (divergent) shift operator. This is easily seen from the action of the connection representation [16] scalar constraint [smeared with a test density $f(x)$] on a loop state:

$$\hat{C}_{\rm gr}(f)\,\Psi_\alpha[A] = \int d^3f(x){\rm Tr}\left(F_{ab}\frac{\delta}{\delta A_a}\frac{\delta}{\delta A_d}\right){\rm Tr}e^{\oint_\alpha A} = \int dt\int ds\, f(\alpha(s))\,\delta^3(\alpha(s),\alpha(t))\;\dot{\alpha}^a(s)\frac{\delta}{\delta\alpha^a(s)}\Psi_\alpha[A]. \tag{19}$$

The operator in the last line shifts the loop along its own tangent (as well as along the tangent of any intersecting line, if any). Now, consider this *same* shift action on an *open* loop state, where the end points are given by fermion fields, and repeat the above calculation backward,

$$\int dt \int ds f(\alpha(s))\delta(\alpha(s),\alpha(t))\dot{\alpha}^a(s)\frac{\delta}{\delta\alpha^a(s)}(\psi^A(\alpha_i)U_{\alpha A}{}^B\psi_B(\alpha_f))$$
$$= \int d^3x f\left[\mathrm{Tr}\left(F_{ab}\frac{\delta}{\delta A_a}\frac{\delta}{\delta A_d}\right) + 2\dot{\alpha}^a\mathcal{D}_a\psi_A(x)\frac{\delta}{\delta\psi_A(x)}\right]\Psi_\alpha[A], \tag{20}$$

the last term being necessary to shift the fermions sited at the end points. The operator on the last line can be written as

$$\int d^3x\ f\left[\mathrm{Tr}\left(F_{ab}\frac{\delta}{\delta A_a}\frac{\delta}{\delta A_d}\right) + 2\frac{\delta}{\delta A_a}\frac{\delta}{\delta\psi}\mathcal{D}_a\psi\right] \tag{21}$$

and *its classical limit is precisely the Hamiltonian constraint* (3) *of the interacting Einstein-Weyl theory*. To put it dramatically, we might say that by quantizing the *matter-free Einstein equations* in the loop representation, extending the dynamics (the shift operator) to open loops (where the end points are interpreted as spinors), and taking the classical limit, one could have discovered the Dirac equation.

The formalism we have constructed holds only within the "clock regime" [2] in which the coordinate time derivative of $T(x,t)$ is positive. The effective Hamiltonian H becomes imaginary when the system exits this regime. If the formalism is consistent the (real eigenvalue) eigenstates of the Hamiltonian $\hat{H}$ should form an orthogonal basis. This could be a futher step towards the (still open) problem of characterizing the physical scalar product. The difficulty we have left unsolved concerns the determination of the square root. The next step in the present direction of investigation should be to compute matrix elements of $\hat{M}_i$ and $\hat{F}_e$ explicitly, and understand whether there is a direct algorithm for extracting this square root. If this could be done, the theory would be at the stage in which physical evolution of physical states could be described.

We thank Lee Smolin for ideas and encouragement. This work was partially supported by NSF Grant No. PHY-9311465.

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