Extended Loops: A New Arena for Nonperturbative Quantum Gravity

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We propose a new representation for gauge theories and quantum gravity. Alternatively, it can be viewed as a new framework for doing computations in the loop representation. It is based on the use of a novel mathematical structure that extends the group of loops into a Lie group. It puts in a precise framework some of the regularization problems of the loop representation. Making use of it we are able to find a new solution to the Wheeler-DeWitt equation that reinforces the conjecture that the Jones polynomial is a quantum state of nonperturbative quantum gravity.

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The introduction of the loop representation has opened a new avenue for the nonperturbative canonical quantization of general relativity. In particular it allows one to immediately code the invariance under spatial diffeomorphisms of wave functions in the requirement of knot invariance [1]. Also for the first time a large class of solutions to the Wheeler-DeWitt equation has been found in terms of nonintersecting knot invariants. It is not clear, however, how to make these solutions correspond to nondegenerate metrics [2] (a possible solution is the idea of "weaves"; see [3]). Another alternative is to consider knot invariants of intersecting loops and solve the Wheeler-DeWitt equation in loop space [4]. Solutions of this kind have actually been found [5] and they led to the conjecture that the Jones polynomial may be a state of quantum gravity [6].

The definition of many of these states is complicated by regularization ambiguities. Since loops are one dimensional objects living in a three dimensional manifold, they naturally lead to the appearance of distributional expressions. In particular the few knot invariants for which we have analytic expressions require the introduction of regularizations (framings) in the case of intersecting knots. Some invariants even require a regularization for smooth loops [7,8].

This difficulty not only arises for the gravitational case. In non-Abelian gauge theories [9,10] it is known that the quantum states in the loop representation are ill defined. A similar behavior arises [11] even in the simple case of Maxwell theory if one studies the theory in terms of a real connection [12]. Notice that this problem appears *in addition* to the problems of regularization of operators and constraints. In the case of quantum gravity these latter issues are complicated by the requirement of diffeomorphism invariance (for discussion of these issues in the loop framework, see [13,14]).

Loops (sometimes called "hoops" [15], for holonomic loops) are classes of closed curves that give the same holonomy for any gauge connection. They form a group under composition, called the group of loops [16]. This group is not a Lie group, since the composition of loops is only defined for an integer number of loops. Recently, a completion of this group into a Lie group was introduced [17]. The group of loops is included in this Lie group and can be thought of as a discrete subgroup. The elements of the Lie groups are called "extended loops" and are the mathematical basis of the new representation we propose.

Let us discuss for a moment these ideas in the more familiar context of Maxwell theory. The point of departure to construct a usual loop representation is to consider the Wilson loop functional, $W_A(\gamma) = \exp \oint_\gamma dy^a A_a(y)$, where γ is a loop.

Since Wilson loops form an (over)complete basis of gauge invariant functions one can express the wave functions $\Psi[A]$ in terms of Wilson loops and go to a representation purely in terms of loops via the loop transform,

$$\Psi(\gamma) = \int DA W_A(\gamma)\Psi[A]. \qquad (1)$$

The wave functions $\Psi(\gamma)$ are in the loop representation and in this representation one can realize the gauge invariant operators of physical interest of the theory, for instance the Hamiltonian, as was discussed in Ref. [11]. As in other representations, the Hamiltonian needs to be regularized. If one computes the vacuum, one gets

$$\Psi_0(\gamma) = \exp\left(\oint dx^a \oint dy^b D_{ab}(x-y)\right), \qquad (2)$$

where $D_{ab}(x-y)$ is the spatial restriction of the Feynman propagator. This quantity is also ill defined due to the divergence of the propagator when $x = y$. So we have a representation where *both operators and wave functions* are ill defined [if one introduces a regularized propagator, the expression (2) is the vacuum of the regularized theory which, however, may not correspond to the Fock representation].

Let us now consider the quantities $W_A[X] = \exp \int d^3y X^a(y) A_a(y)$. If $X^a(y)$ is a divergence-free vector density the $W_A[X]$'s are gauge invariant. The X's are the Abelian analogs of the extended loops we will consider later on. If one uses $W_A[X]$ instead of the Wilson loop functional in the transform (1), one ends with a representation in which wave functions are functionals of transverse vector densities. It is easy to check that this just corresponds to the electric field representation of the canonically quantized Maxwell field. The Hamiltonian is well known and still needs to be regularized. The vacuum of the theory is simply given by

$$\Psi_0(X) = \exp\left(\int d^3y \int d^3z X^a(y) X^b(z) D_{ab}(y-z)\right), \tag{3}$$

and is well defined (if one restricts to smooth X's) without the need of a regularization.

We therefore see that by extending the idea of loop we have several advantages: on the one hand we end up with a usual representation in terms of fields (it is just the electric field representation); on the other hand because we are using fields instead of distributional objects (loops) the regularization difficulties associated with the wave functions of the loop representation disappear. One can introduce a particular inner product by the functional integration over the X's [11] and the usual Fock structure and the interpretation of the excited states in terms of photons can be recovered completely. It should be emphasized that this does not imply that the use of extended loops bounds us to end in the Fock representation.

The intention of this Letter is to outline a generalization of this procedure to the non-Abelian case, concentrating on the case of gravity.

Let us start by proposing an extension of the notion of holonomy for a non-Abelian field similar to the one we introduced for Maxwell theory. To this aim we rewrite the usual expression for the holonomy as

$$U_A(\gamma) = P \exp\left(\oint_\gamma A_a\, dy^a\right) = 1 + \sum_{n=1}^{\infty} \int dx_1^3 \cdots dx_n^3 A_{a_1}(x_1) \cdots A_{a_n}(x_n) X^{a_1 x_1 \cdots a_n x_n}(\gamma), \tag{4}$$

where γ is a loop and the "multitangents" X are defined by

$$X^{a_1 x_1 \cdots a_n x_n}(\gamma) = \oint_\gamma dy_n^{a_n} \int_0^{y_n} dy_{n-1}^{a_{n-1}} \cdots \int_0^{y_2} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1). \tag{5}$$

The advantage of rewriting the holonomy in this way is that we have captured all the loop dependent information in the multitangents, which are multitensor densities on the spatial manifold. The holonomy therefore can be written in a very economical fashion as the contraction $U_A(\gamma) = A_{\tilde\mu} X^{\tilde\mu}$ where the indices $\tilde\mu$ are a shorthand for $a_1 x_1 \cdots a_n x_n$ and we assume a "generalized Einstein convention" in which we sum from one to three for each repeated index a_i and we integrate over the three manifold for each repeated x_i. A repeated index with a tilde also involves a summation from $n = 0$ to infinity.

The key observation is to notice that if one substitutes in (4) a multitensor density $X^{a_1 x_1 \cdots a_n x_n}$ (not necessarily associated with a loop) such that

$$\partial_{a_i} X^{a_1 x_1 \cdots a_i x_i \cdots a_n x_n} = [\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})] X^{a_1 x_1 \cdots a_{i-1} x_{i-1} a_{i+1} x_{i+1} \cdots a_n x_n}, \tag{6}$$

the trace of the resulting "extended holonomy" $U_A[X]$ is (formally) gauge invariant under gauge transformations connected with the identity. We will call it the extended Wilson functional $W_A[X] = \mathrm{Tr}(U_A[X])$. An important difference is that the use of extended loops allows one to consider quantities that are not invariant under gauge transformations not connected with the identity. Therefore the variables X are able to capture more information than usual loop variables, in particular, information of topological nature.

As we mentioned before, loops form a group. The quantities X can also be endowed with a group structure, and the presence of the extra elements (the ones not associated with loops) allow one to extend the group of loops into a Lie group called the "extended group of loops" [17]. The usual group of loops is a subgroup naturally associated with the multitangents $X(\gamma)$. The group product between multitensors is $(X_1 \times X_2)^{a_1 x_1 \cdots a_n x_n} = \sum_{k=0}^{n} X_1^{a_1 x_1 \cdots a_k x_k} X_2^{a_{k+1} x_{k+1} \cdots a_n x_n}$ and it naturally reproduces loop composition $X(\gamma_1 \circ \gamma_2) = X(\gamma_1) \times X(\gamma_2)$.

The extended Wilson functionals satisfy a series of identities associated with the fact that the gauge theory is associated with a particular gauge group [in the case of gravity SU(2)], which can be explicitly written. In the case of ordinary loops these are the well known Mandelstam identities. Some observations should be made about the resulting space of wave functionals. First of all, they are functionals of the "infinite tower" of multitensors of all orders. Moreover they are *linear* functionals (since the extended Wilson loop is linear in the multitensors). Wave functions must also satisfy Mandelstam identities.

Up to the moment the discussion has been general, in the sense that it could apply as well to any gauge theory. We will now particularize to the case of quantum gravity written in terms of Ashtekar's new variables [18],

$$\hat{C}_b \Psi[A] = \hat{E}_i^a \hat{F}_{ab}^i \Psi[A] = 0, \tag{7}$$

$$\hat{\mathcal{H}} \Psi[A] = \epsilon_{ijk} \hat{E}_i^a \hat{E}_j^b \hat{F}_{ab}^k \Psi[A] = 0, \tag{8}$$

where E_i^a is a triad, A_a^j is the Sen connection, and F_{ab}^i is the curvature. The first set of equations is the diffeomorphism constraint which says that the theory is invariant under diffeomorphisms of the three manifold and the last equation is the Hamiltonian constraint, which corresponds to the Wheeler-DeWitt equation of the usual canonical formulation.

In order to write these equations in the extended representation we choose a polarization in which $\hat{A}$ is multiplicative and $\hat{E}$ a functional derivative. We then consider the action of the elementary operators on extended holonomies and rewrite them in terms of the X's, exactly as one proceeded with loops [1]. From there one can obtain the action of any gauge invariant operator in the extended representation. An important fact is that due to the linearity of *all* wave functions in the extended representation, any gauge invariant operator can only be a *first order* differential operator. In particular, the form of the constraints is [19]

$$\hat{\mathcal{C}}_a(x)\Psi[R] = [\mathcal{F}_{ab}(x) \times R^{bx}]^{\tilde{\mu}} \frac{\delta}{\delta R^{\tilde{\mu}}}\Psi[R], \tag{9}$$

$$\hat{\mathcal{H}}\Psi[R] = [\mathcal{F}_{ab}(x) \times R^{ax\,\overline{bx}}]^{\tilde{\mu}} \frac{\delta}{\delta R^{\tilde{\mu}}}\Psi[R], \tag{10}$$

where if $\tilde{\mu} = (\mu_1 \cdots \mu_n)$ with $\mu_i = a_i x_i$, then $R^{\mu_1\cdots\mu_n} = \frac{1}{2}[X^{\mu_1\cdots\mu_n} + (-1)^n X^{\mu_n\cdots\mu_1}]$, $(R^{bx})^{\mu_1\cdots\mu_n} = \sum_{k=0}^{n} R^{\mu_{k+1}\cdots\mu_n\, bx\, \mu_1\cdots\mu_k}$, and $\mathcal{F}_{ab}$ is an element of the loop algebra such that the field tensor is given in terms of the connection as $F_{ab} = \mathcal{F}_{ab}^{\tilde{\mu}} A_{\tilde{\mu}}$. An explicit form for $\mathcal{F}_{ab}$ can be easily written and the only nonvanishing components are of the first and second rank. $(R^{ax\,\overline{bx}})^{\mu_1\cdots\mu_n} = \sum_{k=0}^{n}(-1)^{n-k} R_c^{ax\,\mu_1\cdots\mu_k\, bx\, \mu_n\cdots\mu_{k+1}}$ and the inversion in the order of the last set of indices in this definition has a role totally analogous to the "reroutings" at the intersections of loops of the traditional Hamiltonian constraint in the loop representation. The subindex c stands for the cyclic combination in upper indices. Because of the Maldestam identities the wave functions depend only on the combination R, the "even" part of the coordinates X. One can particularize the above expressions to the case when the multitensors are the multitangents to a loop. The resulting expressions *correspond to the usual constraints of quantum gravity in the loop representation* [4,20].

What about solutions to the constraints? It should be pointed out that since loops are a particular case of multitensors, *any* solution found in terms of multitensors can be particularized to loops and would yield a solution to the usual constraints of quantum gravity in the loop representation (the particularization could in some cases yield a singular function, for instance when loop expressions need to be framed, as in the case of regular isotopic knot invariants [7,8]). It should be emphasized that the converse is not necessarily true: given a solution in the loop representation, it may not generalize to a solution in the extended representation. An immediate example is the solutions to the Hamiltonian based on smooth nonintersecting loops, which find no immediate analog in the extended representation.

It is therefore remarkable that there actually exists a particular family of solutions in the loop representation which *does* generalize to the extended representation. In the connection representation, the exponential of the Chern-Simons form built with the Ashtekar connection is a solution of all the constraints of quantum gravity with a cosmological constant [21,22]. When transformed into the loop representation, the resulting wave function is the Kauffman Bracket knot polynomial, which is a phase factor times the Jones polynomial. Through a close examination, it was conjectured that the coefficients of the Jones polynomial are solutions of the Hamiltonian constraint without cosmological constant. Evaluating the loop transform using perturbative techniques of Chern-Simons theory [8], explicit expressions for the Kauffman Bracket (KB) coefficients can be found,

$$\mathrm{KB}_\Lambda(\gamma) = e^{\Lambda GL(\gamma)}[1 + a_2(\gamma)\Lambda + a_3(\gamma)\Lambda^3 + \cdots], \tag{11}$$

where $GL(\gamma) = g_{ax\,by}X^{ax}(\gamma)X^{by}(\gamma)$ is the Gauss self-linking number $[g_{ax\,by} = \epsilon_{abc}(x-y)^c/|x-y|^3$ is the free propagator of Chern-Simons theory]. $a_2(\gamma)$ and $a_3(\gamma)$ are coefficients of an expansion of the Jones polynomial evaluated in $\exp(\Lambda)$ that can be explicitly written as linear functions of the multitangents with coefficients constructed from $g_{ax\,by}$. Through a laborious computation it was shown that $a_2(\gamma)$ satisfied the Hamiltonian constraint of quantum gravity [5] and it was later conjectured [6] that similar results may hold for $a_3(\gamma)$ and higher coefficients.

Analogous expressions can be written in the extended representation simply by replacing the multitangents that appear in the definition of the coefficients with arbitrary multitensors. It can be checked that the resulting expressions are diffeomorphism invariant and no framing problems arise. Remarkably, *they solve the constraint equations in the extended representation.* Moreover, due to the simplification of the manageability of the constraint a new solution can be found. The recently obtained expression [23] for the third coefficient of the Jones polynomial, a_3, *can also be checked to be a solution of the Wheeler-DeWitt equation* [24] in the extended representation. This adds more credibility to the conjecture that the Jones polynomial could be a state of quantum gravity.

Generically, the multitensors are distributional, as can be immediately seen from the equation they satisfy (6). However, their distributional character is under control. As was shown in Ref. [17] a generic multitensor satisfying (6) can be written as a linear combination of transverse multitensors (which one can restrict to be smooth) times some well defined distributional coefficients. This has important consequences. In particular the solutions considered above are such that the dependence on the distributional coefficients drops off and they are only functions of the smooth part of the multitensors. They are therefore well defined. This was not the case in the loop representation, where they needed to be framed to be well defined. This is because in order to recover the usual loop representation one needs to choose the transverse part of the multitensors to be distributional. Once one has chosen to work with smooth transverse multitensors, no further restriction is required for these wave functions to solve the constraints (recall that in terms of loops sometimes other restrictions, like conditions on intersections, were required for certain wave functions to solve the constraints). As a consequence, if one formally particularizes these states to the case of loops, they solve the constraints for an arbitrary type of intersection.

The fact that the wave functions are well defined without regularization ambiguities does not directly imply that the operators in this representation are well defined. As in any functional representation, the presence of functional derivatives may introduce singularities that need to be regularized and renormalized. In the extended representation, a regularization problem is present in the Hamiltonian constraint which involves a multitensor with a repeated spatial dependence [as can be seen from (6), repeating a spatial dependence involves a singularity]. However, because the singular nature of the multitensors is under control one can perform a precise point-splitting regularization and it can be checked in detail if the coefficients of the extended Jones polynomial presented above are annihilated by the regularized constraints or not. This issue is currently being studied.

One can view the role of the multitensors in the extended representation as configuration space variables of a canonical theory. The conjugate momenta are represented by functional derivatives. This suggests that there exists an underlying classical Hamiltonian theory that under canonical quantization yields directly the extended loop representation. This was unclear with loops, where the loop representation could only be introduced through a noncanonical quantization. For the Maxwell case this theory was studied [25] and found to be equivalent to the usual Maxwell theory.

Two last points are in order. First, it is not clear if the extended loops considered in this paper are way too overcomplete to be an adequate description of quantum gravity. It may be that when a clearer understanding is gained, a more modest extension of the concept of loops will be sufficient to address the issues we have discussed in this paper. In particular, the issue of the convergence of extended holonomies is yet to be carefully studied. Apart from this, one of the most attractive ingredients of the loop representation, the fact that the diffeomorphism constraint was easily solvable in terms of knots, is lost. It may happen that a more careful study of the diffeomorphism constraint in terms of extended loops will ultimately reveal some simple characterization of diffeomorphism invariant classes of extended loops, but the issue requires further study.

Summarizing, the extended representation presents practical calculational advantages and offers new possibilities to discuss regularization issues in the theory and set it in a more controlled computational framework without losing some of the topological and geometric insights of the loop representation.

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[1] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990).
[2] B. Brügmann and J. Pullin, Nucl. Phys. **B363**, 221 (1991).
[3] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. **69**, 237 (1992).
[4] R. Gambini, Phys. Lett. B **255**, 180 (1991).
[5] B. Brügmann, R. Gambini, and J. Pullin, Phys. Rev. Lett. **68**, 431 (1992).
[6] B. Brügmann, R. Gambini, and J. Pullin, General Relative Gravity **25**, 1 (1993).
[7] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[8] E. Guadagnini, M. Martellini, and M. Mintchev, Nucl. Phys. **B330**, 575 (1990).
[9] Yu. M. Makeenko and A. A. Migdal, Phys. Lett. **88B**, 135 (1979).
[10] R. Gambini and A. Trias, Nucl. Phys. **B278**, 436 (1986).
[11] C. Di Bartolo, F. Nori, R. Gambini, and A. Trias, Lett. Nuovo Cimento Soc. Ital. Fis. **38**, 497 (1983).
[12] It is possible to construct many nonequivalent loop representations for the Maxwell case, depending on what connection one takes as a fundamental variable. The difficulty discussed here appears when one takes the usual connection, which is the only construction that naturally generalizes to Yang-Mills theory. One can also take the (complex) self-dual connection or a (complex) positive frequency connection; A. Ashtekar and C. Rovelli, Classical Quantum Gravity **9**, 1121 (1992).
[13] C. Rovelli, Nucl. Phys. **B405**, 797 (1993).
[14] L. Smolin, Phys. Rev. D **49**, 4028 (1994).
[15] A. Ashtekar and J. Lewandowsky, in "Knots and Quantum Gravity," edited by J. Baez (Oxford University Press, Oxford, to be published).
[16] R. Gambini and A. Trias, Phys. Rev. D **23**, 553 (1981).
[17] C. Di Bartolo, R. Gambini, and J. Griego, Commun. Math. Phys. **158**, 217–240 (1993).
[18] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987).
[19] C. Di Bartolo, R. Gambini, and J. Griego (to be published).
[20] B. Brügmann and J. Pullin, Nucl. Phys. **B390**, 399 (1993).
[21] H. Kodama, Phys. Rev. D **42**, 2548 (1990).
[22] B. Brügmann, R. Gambini, and J. Pullin, Nucl. Phys. **B385**, 581 (1992).
[23] C. Di Bartolo and J. Griego, Phys. Lett. B **317**, 540 (1993).
[24] C. Di Bartolo and J. Griego (to be published).
[25] D. Armand-Ugon, R. Gambini, J. Griego, and L. Setaro, Report No. hep-th:9307179, 1993 (to be published).