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Haldane Exclusion Statistics and Second Virial Coefficient

M. V. N. Murthy and R. Shankar
Institute of Mathematical Sciences, Madras 600 113, India
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We show that Haldane's new definition of statistics, when generalized to infinite dimensional Hilbert spaces, is determined by the high temperature limit of the second virial coefficient. We thus show that this exclusion statistics parameter g of anyons is nontrivial and is completely determined by its exchange statistics parameter α. We also compute g for quasiparticles in the Luttinger model and show that it is equal to α.

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There has been much interest in the physics of anyons in the past few years. Anyons are particles whose many particle wave functions pick up a phase $e^{i\alpha\pi}$ under the exchange of the positions of any two particles [1]. Arbitrary values of α are allowed by the configuration space topology of many particle systems in one and two spatial dimensions. In particular $\alpha = 0$ corresponds to completely symmetric wave functions (bosons) and $\alpha = 1$ corresponds to antisymmetric wave functions (fermions). The parameter α is traditionally called the statistics of the particle, since for $\alpha = 1$, the antisymmetry of the wave function implies the Pauli exclusion principle. This has nontrivial consequences for the counting of states in the many particle system and hence on the statistical mechanics.

The effect of the exchange phase on the exclusion principle for arbitrary α was largely unexplored until recently. In a seminal paper [2], Haldane has proposed an alternate definition of statistics based on a generalized exclusion principle. His definition is for particles with finite dimensional Hilbert spaces but can, in principle, lead to the existence of fractional statistics in arbitrary spatial dimensions. The parameter governing fractional statistics in this case is defined by $g = -\frac{d_{N+\Delta N} - d_N}{\Delta N}$, where N is the number of particles and d_N is the dimension of the one particle subspace in N-particle Hilbert space. By definition, therefore, $g = 0$ corresponds to bosons and $g = 1$ to fermions. Haldane applied this definition to two cases: quasiparticles in fractional quantum Hall effect (FQHE) systems and spinons in quantum antiferromagnets. In both cases he argued that the exclusion statistics parameter g is equal to the exchange statistics parameter α. Johnson and Canright [3] have tested this prediction numerically for FQHE systems and find this to be true with the modification that for quasiparticles, $g = 2 - \frac{1}{3}$ whereas $\alpha = -\frac{1}{3}$.

In this paper we first generalize Haldane's definition to the case where the Hilbert space of the particles is infinite dimensional. Our motivation for doing so is to apply his concept of the fractional exclusion principle to systems of particles in the continuum, when there is no natural sharp cutoff (which is the case when there are no large gaps in the spectrum). We then show that in general the exclusion statistics parameter g is determined by the dimensionless second virial coefficient provided the system admits a virial expansion of the equation of state in the high temperature limit. We apply this result to the case of anyon gas and find that $g = 2\alpha - \alpha^2$, when the exchange statistics parameter α is chosen to be in the range $0 \leq \alpha \leq 2$. We then analyze the case of anyons in a magnetic field, confined to the first Landau level. We obtain $g = \alpha$ for this case. We also apply our result to the case of quasiparticles in a one component Luttinger liquid. These quasiparticles also have a nontrivial exchange phase due to the braiding properties of the vertex operators that create them. Here we also find that $g = \alpha$; the spinons are a special case of this when $\alpha = \frac{1}{2}$. Note that for quasiparticles with exchange statistics $-\frac{1}{3}$ we have to take $\alpha = 2 - \frac{1}{3}$. Thus our definition and calculation are consistent with the numerical findings [3].

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The dimension of the Hilbert space of N identical particles when g is the statistics parameter is given by [2,3]

$$D_N(g) = \frac{[d + (1-g)(N-1)]!}{N![d-1-g(N-1)]!}, \tag{1}$$

where d is the dimension of the single particle space. It is easy to see that $g = 0$ (1) yields the dimension of the bosonic (fermionic) Hilbert space. It is now straightforward to extract the statistics parameter g from Eq. (1); in the limit where the single particle dimension $d \to \infty$, we obtain

$$\frac{1}{2} - g = \lim_{d\to\infty} \frac{d}{N(N-1)} \left[N! \frac{D_N(g)}{d^N} - 1 \right]. \tag{2}$$

Note that in this limit N is finite and $N/d \to 0$. This, however, is consistent with the density of particles $\rho = N/V$ being finite since for the particles in the continuum $d/V \to \infty$. A regulated definition of the dimension of the Hilbert space is given by the corresponding N-particle partition function since we have $D_N = \lim_{\beta\to 0} Z_N = \lim_{\beta\to 0} \mathrm{Tr}(e^{-\beta H_N})$, where β is the inverse temperature and H_N denotes the N-particle Hamiltonian. Therefore, for dealing with infinite dimensional Hilbert spaces, we propose a generalization of Haldane's definition,

$$\frac{1}{2} - g = \lim_{\beta\to 0} \frac{C Z_1}{N(N-1)} \left(N! \frac{Z_N}{Z_1^N} - 1 \right), \tag{3}$$

where C is an overall constant of proportionality which we fix next. The definition of g given by Eq. (3) forms the basis for the rest of the calculations presented in this paper.

To test the utility of Eq. (3), we first consider the case of bosons and fermions in d_s space dimensions. For convenience we confine all the N particles in an oscillator potential of the form $V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \frac{1}{2} m\omega^2 \sum_{i=1}^{\infty} r_i^2$, where ω is the oscillator frequency and $\mathbf{r}_i$ are single particle coordinates. The potential here merely acts as a regulator and the oscillator potential which will be used throughout this paper is only a convenient choice. The N-particle partition function is then given by the expansion [4]

$$Z_N^{B,F}(\beta) = \frac{1}{N!} \left[Z_1^N(\beta) \pm Z_1(2\beta)^{N-2} \frac{N(N-1)}{2} + \cdots \right], \tag{4}$$

where the $\pm$ signs refer to bosons and fermions, respectively. Notice that the second term in Eq. (4) involves $Z_1(2\beta)$ since the contribution to this term comes when two particles are in the same energy level. Equation (4) is an obvious generalization of Eq. (1) with a cutoff on the single particle energies. The single particle partition function is given by

$$Z_1(\beta) = \frac{e^{-\beta\omega d_s/2}}{(1 - e^{-\beta\omega})^{d_s}}. \tag{5}$$

Substituting Eq. (5) and Eq. (4) in Eq. (3) and taking the high temperature limit we find $\frac{1}{2} - g = \pm \frac{C}{2(2)^{d_s}}$ which essentially isolates the contribution of the subleading order term in Eq. (4). We therefore set $C = 2^{d_s}$ and immediately obtain $g = 0$ (1) for bosons (fermions). Thus factor C occurs because the regulated definition cuts off the energy at 2β when two particles occupy the same state; however, for counting purposes we want to cut off all states at the same energy, irrespective of their occupation.

We are now in a position to apply the definition of g as generalized to infinite dimensional Hilbert spaces by Eq. (3) to N-particle systems with interactions. To begin with, consider a system of N-interacting bosons in two space dimensions. We first show the connection between Eq. (3) and the second virial coefficient in the equation of state of the system. The virial expansion for the pressure P of a gas is by definition a high temperature or low density expansion in terms of the particle number density $\rho = N/V$, where N is the particle number and V is the volume (area in two space dimensions). In the thermodynamic limit $N \to \infty$ and $V \to \infty$ when ρ is held fixed [5,6]. The virial expansion is then given by

$$\beta P = \rho \left[1 + \sum_{l=1}^{\infty} B_{l+1} (\rho\lambda^2)^l \right], \tag{6}$$

where $\lambda = \sqrt{2\pi\beta/m}$ is the thermal wavelength and the dimensionless B_l's are called virial coefficients which can be expressed in terms of the partition functions. For example, the second virial coefficient B_2 is given by

$$B_2 = Z_1 \left[1 - 2\frac{Z_2}{Z_1^2} \right]. \tag{7}$$

(Note: Conventionally B_2 has V/λ^2 as an overall factor instead of Z_1 when box normalization is used.) Comparing Eq. (7) and Eq. (3) we immediately obtain g for the case $N = 2$,

$$\tfrac{1}{2} - g = -2B_2. \tag{8}$$

This is an exact result for $N = 2$. Now consider $N \geq 3$. The higher virial coefficients are given by

$$B_3 = 4B_2^2 - 2b_3, \qquad B_4 = 9B_2B_3 - 16B_2^3 + 3b_4. \tag{9}$$

In general we have for $k \geq 3$

$$B_k = F_k(B_2, B_3, \ldots, B_{k-1}) + (-1)^{k-1}(k-1)b_k, \tag{10}$$

where F_k are some functions of $B_2, B_3, \ldots, B_{k-1}$ whose explicit form is not needed here and b_k is given by

$$b_k = (Z_1)^{k-1}(-1)^{k-1} \sum_{\{m_i\}} (-1)^{(\sum_i m_i - 1)} \left(\sum_i m_i - 1 \right)! \prod_i \frac{(Z_i/Z_1^i)^{m_i}}{m_i!}. \tag{11}$$

The summation over m_i is constrained by $\sum_i^k i m_i = k$. Thus the virial expansion of the equation of state exists if all b_k's are finite. In order to proceed with the computation of g for $N \geq 3$, let us assume that for the system under consideration the virial expansion exists and hence all B_k's are finite. The high temperature expansion for the factor Z_N/Z_1^N in Eq. (3) is in general given by

$$\frac{Z_N}{Z_1^N} = \frac{1}{N!} + f_2^{(N)}(\beta\omega)^2 + \cdots, \tag{12}$$

where $f_2^{(N)}$ is some function of the interaction strength and may depend on N. In the above equation we have only shown the expansion up to next to leading order as $\beta \to 0$; higher order terms are irrelevent for our purposes. Using Eq. (12) we immediately see that the leading divergence in b_k as $\beta \to 0$ comes from the overall factor Z_1^{k-1} and is given by $1/(\beta\omega)^{2(k-1)}$. For b_k to be finite in the limit $\beta \to 0$, we must demand that all terms in the sum involving powers of Z_i/Z_1^i in Eq. (11) up to order $(\beta\omega)^{2(k-1)-1}$ must cancel order by order. This necessarily implies that the coefficient of $(\beta\omega)^2$ must be set to zero. Demanding this we immediately find

$$f_2^{(N)} = \sum_{n=1}^{N-2} \frac{f_2^{(N-n)}}{n!}(-1)^{(n-1)} = N(N-1)\frac{f_2^{(2)}}{N!}, \tag{13}$$

where all $f_2^{(N)}$ are now related to $f_2^{(2)}$ corresponding to the two particle partition function. Substituting Eq. (12) and Eq. (13) in Eq. (3) we obtain for g

$$\tfrac{1}{2} - g = C f_2^{(2)} = -2B_2. \tag{14}$$

We now have, therefore, a relation between g and the second virial coefficient B_2 for a system for which the virial expansion exists. Notice that in obtaining the result (14) for g, it is sufficient to demand that order $(\beta\omega)^2$ terms cancel in the sum given in Eq. (11). This is obviously a much weaker condition than demanding that the virial coefficients be finite. Nevertheless, the whole analysis makes sense only if the virial expansion is valid. This then is the main result of the paper: *The Haldane exclusion statistics parameter g is completely determined by the high temperature limit of the second virial coefficient of a system of interacting particles in the continuum which admits a virial expansion in this limit.*

We can now apply this result to some well known systems. We first consider anyon gas. Here the second virial coefficient is well known and is given by [7]

$$B_2 = -\frac{2\alpha^2 - 4\alpha + 1}{4}, \tag{15}$$

where $\alpha = 0$ is bosonic and $\alpha = 1$ is fermionic. Using Eq. (14) we therefore have

$$g = \alpha(2-\alpha), \tag{16}$$

which has the correct limit for $\alpha = 0, 1$. This result would be true for two anyons. However, it is not conclusively proved that the higher virial coefficients are finite in the case of anyon gas. It has been proved that the third virial coefficient is finite [4,8]. If indeed all virial coefficients are finite, then the result given in (16) would be true for anyon gas in general. This is an interesting result since the exclusion statistics parameter g is not the same as the exchange statistics α, unlike the systems considered by Haldane. For a lattice anyon gas Haldane argued that the particles would be classified as hard-core bosons (hence fermions) since the coupling of a particle to Chern-Simons gauge field does not affect Hilbert space dimensions. Obviously our result shows that this is not so in the continuum. It is easy to see why. The two anyon spectrum, after taking out the center of mass, is given by $E_{n,l} = \omega(2n + |l-\alpha| + 1)$. When α is nonzero all the energy levels are shifted up by α ($l \leq 0$) or $2-\alpha$ ($l > 0$) (equivalently $-\alpha$), no matter where the cutoff lies.

Next consider anyons confined in a magnetic field. Once again g is given by Eq. (16) when the partition function includes the trace over all Landau levels. This is easy to see since both the oscillator frequency ω and the cyclotron frequency ω_c act as regulators and at high temperatures one recovers the anyon gas result. If, however, we confine the trace to a single Landau level, we obtain $g = \alpha$ in conformity with the results obtained for FQHE systems.

We will now consider quasiparticles in the Luttinger model. The low energy physics of this model maps on to the massless Thirring model. As is well known [9], the model is exactly solvable. We will therefore be able to compute the right-hand side (rhs) of Eq. (3) directly in this case. The theory can be written completely in terms of the left (L) and right (R) currents satisfying the algebra $[J_n^r, J_m^{s\dagger}] = n\delta_{mn}\delta_{rs}$, where $n, m = 1, 2, \ldots$ and $r, s = L, R$. We also have the zero modes $[\phi_0^r, J_0^s] = i\delta_{rs}$. The Hamiltonian is

$$H = \frac{2\pi v_F}{L}\sum_r \left[\frac{1}{2}J_{0r}^2 + \sum_{n=1}^{\infty} J_n^r J_n^{r\dagger}\right]. \tag{17}$$

Quasiparticles are created by the action of the vertex operators $V_r^\dagger(x) =: e^{-iq^r\phi^r(x)}$: on the ground state. The $\phi^r(x)$ are the bosonic phase fields given by

$$\phi^r(x) = \phi_0^r(x) + \frac{2\pi}{L}\sigma_r\left[xJ_{0r} + \sum_{n=1}^{\infty}\left(\frac{e^{-i\sigma_r\frac{2\pi}{L}nx}}{in}J_n^r + \text{H.c.}\right)\right], \tag{18}$$

where $\sigma_{L,R} = \pm 1$. The periodicity properties of the compact zero modes $\phi_0^{L,R}$ constrain the allowed values of q_r to be $q_r = NR + \sigma_r\frac{M}{2R}$ and R is the radius of the field ϕ. It is a function of the interaction strength. Our conventions correspond to $R = 1$ for the case of noninteracting fermions. The constraints on q_r imply that quasiparticles in the left sector must be created along with quasiparticles in the right sector (except for special values of R). However, because the Hamiltonians of the two sectors are completely decoupled, it is consistent to analyze the spectrum of the two sectors independently. We will therefore focus on the left sector alone.

We first consider the space of one quasiparticle states which we define to be the space of the states $|x\rangle = V_L^\dagger(x)|0\rangle$. These are not a linearly independent set. From the form of $V_L^\dagger(x)$, it follows that $|x+L\rangle = e^{i\alpha\pi}|x\rangle$, where $\alpha = q_L^2$. Therefore, we have the expansion $|x\rangle = \sum_{n=1}^{\infty} e^{ik_n x}|n\rangle$, where $k_n = \frac{2\pi}{L}(n + \frac{\alpha}{2})$. From the form of $V_L^\dagger(x)|0\rangle$ it follows that $n > 0$. It can be easily shown that $|n\rangle$ form an orthogonal set of eigenstates of the Hamiltonian with eigenvalues $E_n^{(1)} = v_F k_n$. The single particle partition function is then given by

$$Z_1 = \frac{e^{-\tilde{\beta}\alpha/2}}{1 - e^{-\tilde{\beta}}} = e^{-\tilde{\beta}\alpha/2} Z_1^B, \quad \tilde{\beta} = \frac{2\pi}{L}\beta v_F. \tag{19}$$

Next we come to the N-quasiparticle states which we define to be the span of $|\{x_n\}\rangle = \prod_{n=1}^N V_L^\dagger(x_n)|0\rangle$. From the fact that $V_L^\dagger(x)V_L^\dagger(y) = e^{i\alpha\pi}V_L^\dagger(y)V_L^\dagger(x)$, it follows that these many quasiparticle states pick up a phase $e^{i\alpha\pi}$ under the exchange of two of the coordinates. We normal order the vertex operators to obtain $|\{x_n\}\rangle = \prod_{n>m}\left[\sin\left(\frac{\pi(x_n-x_m)}{L}\right)\right]^\alpha : \prod_{n=1}^N V_L^\dagger(x_n) : |0\rangle$. Again from the form of the vertex operators it follows that we have the expansion

$$|\{x_n\}\rangle = \prod_{n>m}^{N}\left[\sin\left(\frac{\pi(x_n - x_m)}{L}\right)\right]^\alpha \times \sum_{\{k_n\}} \phi_N^B(\{k_n\}|\{x_n\})|\{k_n\}\rangle, \tag{20}$$

where $\phi_N^B(\{k_n\}|\{x_n\})$ are symmetrized N-particle plane waves with momenta $\{k_n\}$, where $k_n = \frac{2\pi}{L}(n + \frac{N\alpha}{2})$; $n > 0$. The states $|\{k_n\}\rangle$ can be shown to be eigenstates of the Hamiltonian with energy $E^N(\{k_n\}) = v_F\sum_{n=1}^N k_n$. We can show that $|\{k_n\}\rangle$ are a linearly independent set of states. The states with the same energy are, however, not orthogonal. The N-quasiparticle partition function is then $Z_N = e^{-\tilde{\beta}N^2\alpha/2}Z_N^B$, where Z_N^B is the N-particle bosonic partition function. We can now exactly compute g using the high temperature expansion for Z_N^B and obtain $g = \alpha$. Thus the exchange and exclusion statistics parameters are identical for these models. Note that the exchange statistics in one dimension is somewhat arbitrary. We could have changed it by multiplying the vertex operators with suitable cocycle factors. The exclusion statistics parameter is, however, unambiguous and unique. When $R = \frac{1}{\sqrt{2}}$, the theory is equivalent to the low energy physics of the SU(2) symmetric quantum antiferromagnetic chain. We then have the spinon excitations with $\alpha = \frac{1}{2}$. Thus we recover Haldane's result that $g = \frac{1}{2}$ for this case.

The above example gives a clear insight into the mechanism of the phenomenon. What is happening is that the addition of a quasiparticle causes a phase shift of every other quasiparticle, resulting in an energy shift of $\frac{\alpha\pi v_F}{L}$ per particle . When we count the dimension of the single particle space with a fixed (smooth) cutoff, there are α states missing. The important thing here is that all the single particle levels shift up by the same amount, however high the energy. This is why we get g to be well defined and nontrivial in the cutoff going to infinity limit.

Exactly the same thing happens in the case of the anyon gas except that the relevent levels do not all shift by the same amount. The $l = 0, -2, -4, \ldots$ levels in the two anyon spectrum shift up whereas the positive $l = 2, 4, 6, \ldots$ levels shift down by an amount $-\alpha$. Equivalently, the latter set may be considered as made up of levels $l = 0, 2, 4, 6, \ldots$ with an upward shift given by $2-\alpha$. This is why g is not equal to α but has a nonlinear dependence on it given by the product of the two energy shifts, namely, $\alpha(2-\alpha)$ as though the full g is the product of g's corresponding to two one-component systems. However, the important fact is that g is completely determined by α. When we consider anyons in a strong magnetic field and restrict ourselves to the first Landau level, then again we have l's of only one sign. This again results in $g = \alpha$.

Thus the fundamental character of exclusion statistical interactions seems to be that they cause scale-invariant phase shifts and hence scale-invariant energy shifts. How this is connected to the nonorthogonality of position eigenstates which is stressed in Ref. [2] is not clear to us. We also note that these scale-invariant phase shifts, if they occur in any model, are nonperturbative effects and the model cannot be analyzed by perturbation theory [10]. Thus a nontrivial g implies the inapplicability of standard many-body perturbation theory, as has been stated in Ref. [2].

Finally, we have related g to the high temperature limit of a thermodynamic quantity, which, in principle, is much easier to measure than the exchange phase of quasiparticles in condensed matter systems. It may therefore be possible to use this connection to devise a realistic experiment to measure the Haldane exclusion statistics of quasiparticles.

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