Effects of Additive Noise on On-Off Intermittency

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On-off intermittency is an aperiodic switching between static, or laminar, behavior and bursts of oscillations. It has been recently investigated in a class of one-dimensional maps that are multiplicatively coupled to either random or chaotic signals. The addition of noise to the system results in a fundamental change in the nature of the intermittency. The critical onset parameter changes discontinuously, and the distribution of the laminar phases is modified. These contrasts are examined in terms of random walks.

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There are a large number of physical phenomena exhibiting a peculiar behavior: the system is quiescent for long periods followed by a burst of activity. This behavior is persistent, and can be characterized by intermittent switching of system variables. Examples are abundant in nature and include cyclic starspot activity in astrophysics, intermittent turbulent bursts occurring in otherwise laminar pipe flow in fluid dynamics, irregular reversals of the Earth's magnetic field, and earthquakes.

A simple model of intermittency, so called on-off intermittency, has been recently introduced by Platt, Spiegel, and Tresser [1]. Some features of this type of intermittency have been described by other authors [2]. However, a surprising feature of this model is that the addition of arbitrarily small amplitude "noise" to the model causes a large discontinuous change in the critical onset threshold, and modifies the statistical behavior of the intermittency. In contrast to the paper introducing on-off intermittency [1], this work makes quantitative predictions that should be of interest to anybody who is trying to verify on-off intermittency and its signatures *experimentally.*

The mechanism responsible for the on-off intermittency model is a *dynamic* time-dependent forcing of a bifurcation parameter of some simple dynamical system through a bifurcation point. This contrasts with other models such as Pomeau-Manneville intermittency types I–III [3] and crisis-induced intermittency [4], for which the parameters are *static.* Suppose that the slaved (or dependent) system (A) displays on-off intermittency, and the dynamics of the driving system (B) is independent of the dynamics of the observable system (A). This skew product structure [5] provides a clear picture of the dynamics involved and significantly simplifies the characterization of on-off intermittency. In [6], Heagy, Platt, and Hammel analyzed on-off intermittency in a class of one-dimensional maps which are multiplicatively coupled to either random or chaotic drivings. In this work the conditions for the onset of intermittency are calculated and an expression for the distribution of laminar phases (probability of observing a laminar phase of given length) is derived.

The on-off model of intermittency allows the intermittent signal to get arbitrarily close to a fixed point of the observable slaved system (A) during the laminar phases. This is quite unrealistic in real physical situations when a large number of variables are involved and/or when external noise is present. In this situation the fixed point of the dynamical system is actually only *quasifixed,* namely, the system is fluctuating slightly in a small neighborhood of the fixed point during the laminar phases. This can be modeled by a small amplitude additive noise term in system (A). Recently, Hughes and Proctor [7] investigated a system of three-dimensional ordinary differential equations and observed a striking difference in behavior depending on the presence or absence of additive noise terms. In this paper we will investigate the effects of small amplitude additive noise on the statistical properties of on-off intermittency. The properties we examine are the critical value of the bifurcation parameter for the onset of intermittency and changes in the distribution of the laminar phases as the amplitude of the noise is varied.

Following [6] we study maps of the form

$$y_{n+1} = a x_n f(y_n), \tag{1}$$

with $f(0) = 0$, $\frac{\partial f}{\partial y}|_0 \neq 0$, and a is a bifurcation parameter. Here, x_n comes from a chaotic or a random process and for simplicity we assume that x_n has a uniform distribution in the interval $(0, 1]$. Without loss of generality we can assume $\frac{\partial f}{\partial y}|_0 = 1$. Expanding $f(y)$ in a Taylor series around $y = 0$ one obtains

$$y_{n+1} = a x_n (y_n + O(y_n^2)). \tag{2}$$

The nonlinear terms serve only to keep the solution bounded and are not relevant to the distribution of laminar phases. In actual calculations we use a piecewise linear map which has similar properties to a cubic map. It is defined as

$$y_{n+1} = \begin{cases} a x_n y_n, & \text{if } |y| \leq 0.25 \ , \\ a x_n \frac{1-y_n}{3}, & \text{if } y > 0.25 \ , \\ -a x_n \frac{1+y_n}{3}, & \text{if } y < -0.25 \ . \end{cases} \tag{3}$$

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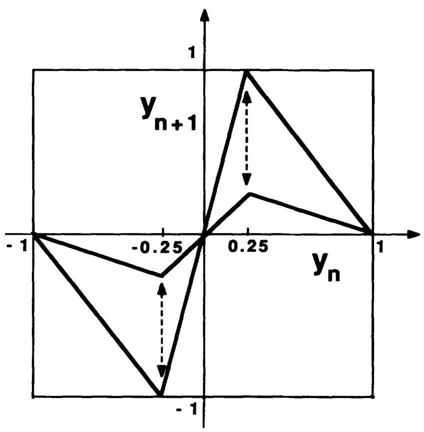

FIG. 1. The first return map y_{n+1} versus y_n of Eq. (3) as the slope factor ax_n is varied.

Figure 1 shows the first return map of (3) as the slope factor ax_n sweeps its range. Note that for $a \leq 4$, the interval $[-1, 1]$ is mapped into itself. Figure 2 shows a typical signal produced by system (3) just beyond the intermittency threshold. To calculate the critical value of the bifurcation parameter $a = a_c$, we note that the solution of (2) keeping terms linear in y_n is $y_n = a^n(\prod_{j=0}^{n-1} x_j)y_0$, and thus the long term behavior of y_n is determined from the asymptotic behavior of the product

$$P_n = a^n \prod_{j=0}^{n-1} x_j. \tag{4}$$

Since $\ln(\prod_{j=0}^{n-1} x_j) = \sum_{j=0}^{n-1} \ln x_j \sim n\langle \ln x \rangle$, and $\langle \ln x \rangle = \int_0^1 \ln x \, dx = -1$, $P_n \sim (\frac{a}{e})^n$. Thus, the critical value of the bifurcation parameter for the onset of intermittent behavior is $a_c = e = 2.718\,28\ldots$.

The distinguishing characteristic of the intermittent signals such as those shown in Fig. 2 is the time between successive "on" events. The distribution of the interevent times, or *laminar phases*, is easily measured (numerically or experimentally), and can serve as a potential classifier of the intermittent signals. Let Λ_n be the probability of obtaining a laminar phase of length exactly n. It is proved in [6] that for system (2) at the onset of intermittency, Λ_n is given by

$$\Lambda_n \sim n^{-\frac{3}{2}} \tag{5}$$

for uniformly distributed random driving. Thus, at the onset of intermittency, the distribution of laminar phases is a power law with exponent $-\frac{3}{2}$. In all cases we have studied so far, the asymptotic distribution of laminar phases at onset is well fit by a power law with exponent $-\frac{3}{2}$. This is true for both random *and* chaotic forcing. Moreover, it is proved in [6] that the $-\frac{3}{2}$ power law is a *universal* feature of on-off intermittency at onset for a large class of *random* driving cases. Figure 3 shows a log-log plot of numerically obtained distributions of laminar phases for various random and chaotic forcings x_n with uniform distribution in the interval $(0, 1]$.

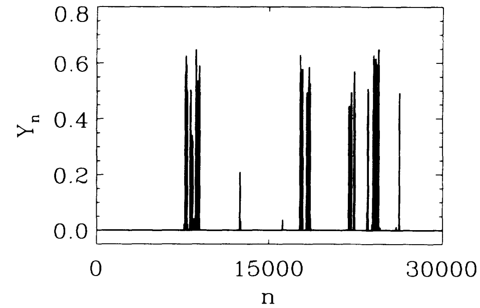

FIG. 2. A typical signal y_n versus time n displaying intermittency at parameter $a = 2.74$.

Computers have a finite numerical precision and some care should be exercised in the numerical calculations of the distribution of laminar phases. To reliably calculate this statistic a very long time series is required, and at the onset of intermittency, $a_c = e$, we expect to encounter laminar phases of arbitrarily long length. In a typical computer realization at $a = 2.72$, signals reach the limit of numerical precision of the computer near zero ($\sim 10^{-320}$) in less than 10^6 iterations, and an underflow condition then rounds off the solution exactly to the fixed point solution of Eq. (2), $y = 0$. To avoid this problem a small amplitude ($\sim 10^{-300}$) additive noise term was used in the calculations of Fig. 3.

In an experimental setting, we cannot expect a system to come arbitrarily close to a laminar fixed point solution. Therefore, it is important to study a modification of Eq. (2), which includes a small amplitude additive noise term,

$$y_{n+1} = ax_n(y_n + O(y_n^2)) + \delta_n 10^{-\nu}. \tag{6}$$

Here, δ_n is a bounded noise process and $\nu > 0$ scales

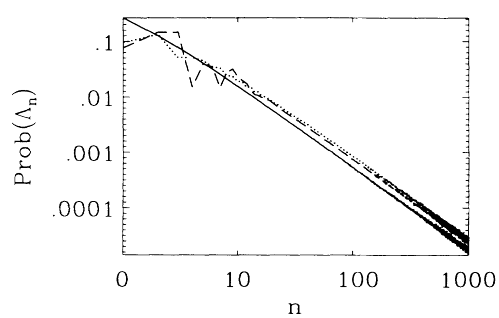

FIG. 3. A numerically calculated log-log plot of the relative probability of a laminar phase of length n versus n at $a = 2.72$. A solid line represents white noise forcing, a dashed line is for tent map forcing, and a dotted line is for $2x \bmod 1$ forcing. All of the lines asymptotically have a slope of $-3/2$.

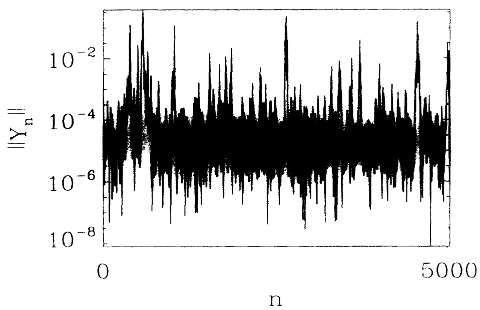

FIG. 4. Typical behavior of the signal for added noise $10^{-\nu}$. The plot shows $\ln|y_n|$ versus n at $a = 2.2$ and $\nu = 5$.

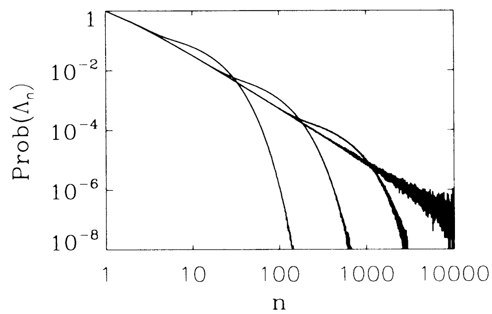

FIG. 5. The numerically calculated log-log plots of the relative probability of a laminar phase of length n versus n at $a = 2.72$, $\tau = 10^{-2}$, and noise levels $\nu = 3$, 5, 10, and 300.

the noise amplitude. For simplicity we assume that $\delta_n \in [-1, 1]$. Note that the distribution function for δ_n is irrelevant to the statistics of the intermittency and in actual calculations δ_n is randomly set to either ± 1. The quantity $10^{-\nu}$ is a *noise level* or a *noise floor*; it represents a boundary below which the dynamics is no longer governed by the noise-free equation (2). We will examine the effect of this modification of Eq. (2) on the critical value of the bifurcation parameter and the distribution of laminar phases.

First, let us examine the mechanism for the intermittency for $a < e$. For $0 < a < e$ the trajectory quickly approaches the noise floor $10^{-\nu}$ and straddles it. Figure 4 shows a plot of the behavior of $\ln|y_n|$ vs n for $a = 2.2$. Most of the time the trajectory remains in a narrow zone around the noise floor with occasional exponential departures. Thus the mechanism for intermittency consists of obtaining a "rare event" — a sequence of x_n which are large on average. This allows the product (4) to grow.

In fact, as shown below, the critical value of the bifurcation parameter is $a_c = 1$ for *any* noise level $10^{-\nu}$. The effect of including additive random noise is equivalent to inserting a zone of radius $10^{-\nu}$ around the fixed point $y = 0$ of Eq. (2). In this zone the dynamics is no longer governed by the linearization in y_n, but instead is governed by random noise. The abrupt change in the critical onset value can be understood by examining the map (2) in the log domain and by treating the added noise as an *elastic barrier* at $y = 10^{-\nu}$ [8]. In this approximation the map (2) becomes an additive random walk with a barrier

$$Y_{n+1} = X_n + Y_n, \tag{7}$$

where $Y_n = \ln|y_n|$ and $X_n = \ln(ax_n)$ with $Y > -\nu$. In the noise-free case the barrier is at $Y = -\infty$. For this case the onset value $a = e$ is equivalent to a walk that is *unbiased*: $\langle X_n \rangle = 0$. Moreover, for any distribution of $x \in (0, 1]$, the critical parameter $a = a_c$ (*onset* value) is determined by the condition $\langle X_n \rangle = 0$. Unbiased random walks are *recurrent* [8]; this is the essential element in the onset of intermittent behavior in the noise-free case. For $a < e$ in the noise-free case the random walk is *biased* (nonrecurrent) and with probability one the walk goes to $Y = -\infty$ $(y = 0)$.

For nonzero noise the walk is bounded below by $Y = -\nu$ and therefore for $a < e$ the walk cannot proceed to $Y = -\infty$. Let $\tau > 10^{-\nu}$ be a threshold above which the signal y_n is considered to be "on." So long as $a > 1$ there is a nonzero probability of starting from the noise level and crossing the threshold. Let $z_n = ax_n$ in Eq. (6) and let $a > 1$ and $\epsilon > 0$ such that $1 + \epsilon < a$. Define an on event E_N, which takes a solution all the way from the noise floor to the threshold τ, by

$$E_N = \{\{z_n\}_{n=0}^{N-1} \mid z_n \in [1 + \epsilon, a]\}. \tag{8}$$

Then $\prod_{n=0}^{N-1} z_n > (1+\epsilon)^N$. Thus, we need to find N such that $(1 + \epsilon)^N 10^{-\nu} > \tau$, namely,

$$N > \frac{\log(\frac{\tau}{10^{-\nu}})}{\log(1 + \epsilon)}. \tag{9}$$

Thus

$$Pr\{E_N\} = \left(\frac{a - 1 - \epsilon}{a}\right)^N > 0, \quad \forall a > 1. \tag{10}$$

Therefore there is a nonzero probability for the event (8). This event can occur in both the noise and noise-free cases. In the noisy case, the event E_N will necessarily result in a return to the threshold since the starting point is fixed. However, in the noise-free case, the starting point will go to $-\infty$ on average. If we assume that the noise δ_n is not correlated with the driving z_n, we deduce that $a_c = 1$ for any noise level $10^{-\nu}$.

We now turn our attention to the distribution of laminar phases at the noise-free critical parameter, $a_c = e$. This distribution changes as a function of noise level $10^{-\nu}$. The distribution of laminar phases at $a = 2.72$ for various noise levels is shown in a log-log plot in Fig. 5. There are two distinguishing characteristics of these plots from those of Fig. 3. The first is the appearance of "shoulders" in the distribution; these shoulders are shifted to larger values of n as the noise level decreases. These shoulders exist for *any* noise level. The second is the exponential falloff in the distribution for large n.

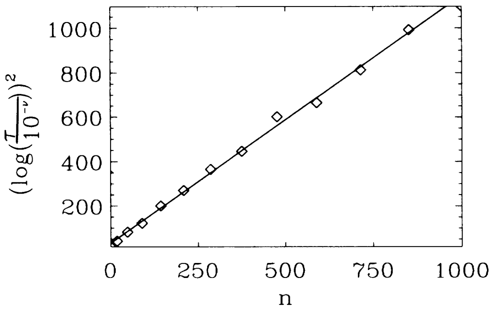

FIG. 6. The scaling of the crossover regions, or shoulders, for various noise levels. Diamonds represent the lengths N^* plotted against the value of $[\ln(\frac{\tau}{10^{-\nu}})]^2$ at $a = 2.72$, $\tau = 10^{-2}$, and at various noise levels ν.

For a given noise level a shoulder represents the crossover time for noise effects to become significant. Let this time be denoted by N^* (a scaling relation for N^* as a function of noise amplitude is given below). For times $n \ll N^*$ the probability of starting at the threshold τ and reaching the noise floor is very small. In other words, the system is "unaware" of the noise floor and the distribution of laminar phases is governed by the noise-free case, namely, a $-\frac{3}{2}$ power law. For times $n \gg N^*$ the majority of paths between successive returns to the threshold τ will have encountered the noise floor at least once and the system is no longer governed by (2). The proof of the exponential decay law for $n \gg N^*$ will be shown in a later work.

From an experimental point of view it is important to know the scaling of the position of the shoulders as the noise level is varied. By the central limit theorem Y_n in the random walk (7) has an asymptotic Gaussian distribution with mean $\mu_n = n\mu$ and variance $\sigma_n^2 = n\sigma^2$, where μ and σ^2 are the mean and variance of the random process $\{X_n\}$. For uniform driving at onset ($a = e$), $\mu = 0$ and $\sigma^2 = 1$, giving $\mu_n = 0$ and $\sigma_n^2 = n$. The crossover time N^* is taken to be the time for the half-width of the distribution (as measured by $\sigma_n/2$) to equal the distance between the threshold and the noise floor. This yields the scaling relation

$$N^* \sim \left[\ln\left(\frac{\tau}{10^{-\nu}}\right)\right]^2. \tag{11}$$

Figure 6 shows the numerically determined crossover time N^* as a function of the quantity $[\ln(\frac{\tau}{10^{-\nu}})]^2$ for several noise strengths. The times N^* were taken to be the points within the shoulders having slope $-\frac{3}{2}$ in Fig. 5. The crossover times agree well with the scaling relation (11).

The examples shown here are not atypical, and the following items should be considered characteristic of on-off intermittency for a wide variety of problems: (a) The $-\frac{3}{2}$ power law in the distribution of laminar phases at onset of intermittency is a universal feature for random drivings when there is no additive noise term. (b) The introduction of additive noise forces a sudden change in the critical onset bifurcation parameter. (c) The distribution of laminar phases is *independent* of the threshold τ in the noise-free case. For the "noisy" intermittency, as the noise level is varied, the distribution of laminar phases is dependent upon the ratio $\frac{\tau}{10^{-\nu}}$.

In addition, we believe that a $-\frac{3}{2}$ power law in the distribution of laminar phases at onset of intermittency holds for a large class of *chaotic drivings* when the driving is completely independent of the responding system.

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