η' Meson Mass in Lattice QCD

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It is shown that the mass difference between η' and pseudoscalar octet mesons can be calculated in quenched lattice QCD with the aid of a variant wall source technique. The estimated mass difference increases as the quark mass decreases, and its value extrapolated to the zero-quark-mass limit. $m_{\eta'}^2 - m_8^2 = (750 \pm 40\ \mathrm{MeV})^2$, is close to the value determined from experiment $(850\ \mathrm{MeV})^2$. These results indicate that the long-standing U(1) problem could be solved in lattice QCD, with its essential part being understood within the quenched approximation. We also comment on implications of our results on spurious infrared divergences in quenched QCD.

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While a substantial understanding has already been achieved concerning the physics of flavor nonsinglet mesons and baryons with the use of lattice QCD, very little has so far been explored for the flavor singlet mesons. The most important issue related to the latter is the U(1) problem that the η' meson mass is much higher than those of the other pseudoscalar octet members [1]. The current view, based on the $1/N_c$ expansion, ascribes the origin of this mass splitting to iteration of virtual quark loops in the η' propagator, each loop giving the factor

$$\frac{m_0^2}{p^2+m_8^2}, \tag{1}$$

where $m_0^2 = m_{\eta'}^2 - m_8^2$ is the η'-octet mass-squared splitting, and to the U(1) anomaly that is supposed to give a large value to m_0 [2]. Whether the U(1) problem is resolved along this view is clearly a problem posed on lattice QCD.

The prime difficulty in exploring the flavor singlet sector with numerical lattice QCD stems from the fact that a calculation of a disconnected two-quark-loop amplitude projected onto the zero-momentum state is prohibitively time consuming, should one attempt to carry it out in a standard manner (for earlier attempts, see Refs. [3,4]). We have shown in a previous work [5] that quite a general class of N-point hadron Green's functions (projected onto the zero-momentum state) is calculable with the use of a variant version of the wall source technique [6]. In this Letter we demonstrate that this technique can be successfully applied to a calculation of the disconnected two-quark-loop amplitude; we have obtained a large mass of the η' meson, which is in a reasonable agreement with experiment, albeit within the quenched approximation and with a relatively large lattice spacing of $a \approx 0.14$ fm at $\beta = 6/g^2 = 5.7$.

The method of calculation of the η'-octet mass splitting comes from the observation that in the quenched approximation the ratio of the disconnected two-quark-loop amplitude to the connected single-quark-loop amplitude for the η' meson propagator, each projected onto the zero-momentum state, should behave as

$$R(t) = \frac{\langle \eta'(t)\eta'(0)\rangle_{2\text{-loop}}}{\langle \eta'(t)\eta'(0)\rangle_{1\text{-loop}}} \approx \frac{m_0^2}{2m_8} t + \text{const} \tag{2}$$

in Euclidean space-time. This formula can be proven by truncating the iteration of the virtual quark loops in the η' propagator, each loop giving the factor of (1) at the two-quark-loop level in agreement with the quenched approximation. The task of lattice QCD then is to extract m_0 directly from the ratio without resorting to any heuristic argument using the U(1) anomaly.

The two-quark-loop amplitude needed in (2) is evaluated as follows. We solve for the quark propagator *with unit source at every space-time site without gauge fixing.* With this quark propagator,

$$G(\mathbf{n},t) = \sum_{(\mathbf{n}'',t'')} G(\mathbf{n},t;\mathbf{n}'',t''), \tag{3}$$

we form the expression

$$\sum_{\mathbf{n}} \mathrm{Tr}[G(\mathbf{n},0)\gamma_5] \sum_{\mathbf{n}'} \mathrm{Tr}[G^\dagger(\mathbf{n}',t)\gamma_5]. \tag{4}$$

This equals the two-quark-loop amplitude, with the two quark loops starting and ending at the time slices $t=0$ and t, up to gauge-variant nonlocal terms which, however, must cancel out in the ensemble average. A very nice feature of this trick is that it requires only a single quark matrix inversion for each gauge configuration in order to calculate the two-quark-loop amplitude for an arbitrary time separation t, whereas in our previous work [3], where we used the conventional point source, we had to invert the quark matrix L^3T times on an $L^3 \times T$ lattice.

We remark that there is no *a priori* guarantee that this method yields a good signal for the propagator, since there are $O((L^3 \cdot L^3 T)^2)$ nonlocal noise terms relative to $O((L^3)^2)$ local gauge-invariant ones in (4), and hence

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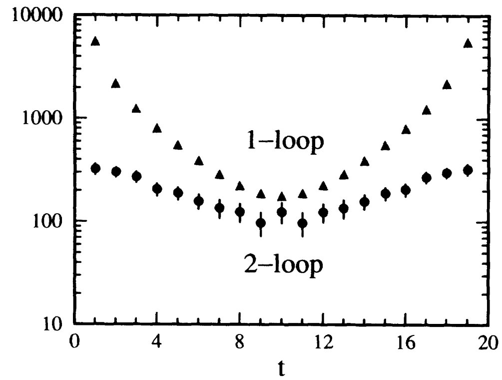

FIG. 1. Two- and single-quark-loop amplitudes for the η' propagator at $\beta=5.7$ and $K=0.1665$ on a $12^3\times 20$ lattice.

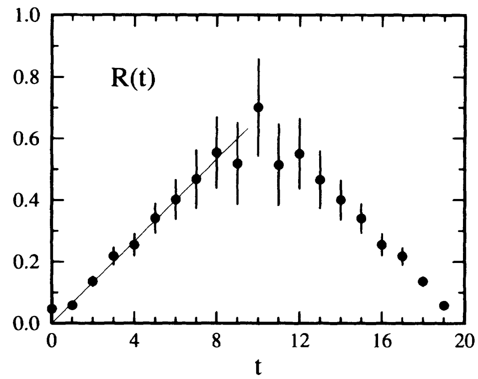

FIG. 2. Ratio $R(t)$ of two- and single-quark-loop contributions to the η' propagator plotted in Fig. 1. Solid line is the linear fit over $4\leq t\leq 8$.

the signal to noise ratio may well be $O(1/T)$. However, in practice we obtain a good signal with reasonable statistics as we shall show below.

Our calculations are made with the Wilson quark action at $\beta=5.7$ with four values of the hopping parameter, $K=0.164$, 0.165, and 0.1665 on a $12^3\times 20$ lattice and $K=0.168$ on a $16^3\times 20$ lattice. For the first three values of K, 240–300 gauge configurations, each 1000 pseudo heat-bath sweeps apart, are used to evaluate the quark propagator, and 43 configurations are employed for the largest K. We encountered three exceptional configurations in the last case. They are excluded from the average, which might induce some bias towards small quark masses. Errors are estimated by the jackknife procedure with 5 configurations as the bin size.

We present in Fig. 1 the two-quark-loop amplitude (circles) of the η' propagator for the case of $K=0.1665$, together with the single-quark-loop amplitude (triangles) for comparison. We observe a good signal for the two-quark-loop amplitude, which demonstrates the effectiveness of our method for calculating disconnected contributions.

The ratio $R(t)$ of the two amplitudes is shown in Fig. 2. A good linear increase with t for the range of $t=1$–8 is seen. While a constant is generally expected in the ratio, the figure shows that it is negligibly small. We then extract m_0 by fitting the data to the linear form (2)

TABLE I. Values of m_0 multiplied by $\sqrt{N_f}=\sqrt{3}$ in lattice units in quenched lattice QCD at $\beta=5.7$ with the Wilson quark action.

K	Lattice size	No. Config.	m_π/m_ρ	m_0
0.164	$12^3\times 20$	300	0.74	0.255(14)
0.165	$12^3\times 20$	240	0.70	0.300(14)
0.1665	$12^3\times 20$	240	0.59	0.379(13)
0.168	$16^3\times 20$	40	0.42	0.452(28)
$K_c=0.1694$				0.518(25)

where the value of m_8 is taken from a standard analysis of the pion propagator, i.e., the single-quark-loop amplitude in the denominator of (2). The fitting range is chosen to be 3–$4\leq t\leq 7$–8, depending on the quality of the data, where the cutoff at small t is made in order to avoid a possible contamination from higher excited states in agreement with the analysis for the pion propagator that indeed exhibits such a contribution. We set the constant term to be zero since its inclusion does not modify the resulting fit, simply increasing errors of the fitted values of m_0. For the case of $K=0.168$ only $t=4$–6 are used for the fit, because of the poor quality of our data.

The results for m_0 are plotted in Fig. 3 as a function of the conventionally defined quark mass $m_q=(1/K-1/K_c)/2$ using $K_c=0.1694$ [7]. The numerical values are given in Table I. Here we multiply m_0 by $\sqrt{N_f}=\sqrt{3}$ to obtain a physical value for the case of three flavors as

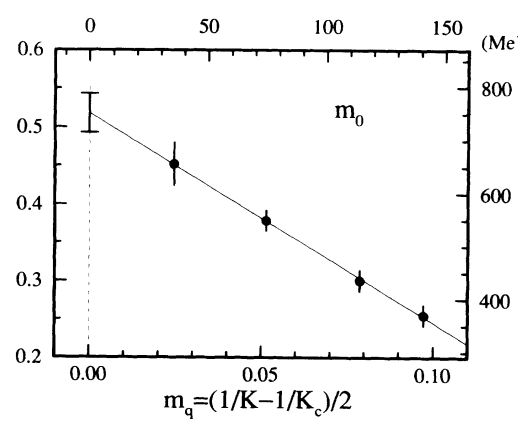

FIG. 3. m_0 multiplied by $\sqrt{N_f}=\sqrt{3}$ as a function of $m_q=(1/K-1/K_c)/2$. Left ordinate and bottom abscissa are in lattice units, while right ordinate and top abscissa are in physical units using $a^{-1}=1.45$ GeV. Solid line is a linear fit which extrapolates to $m_0=0.518(25)$ [751(39) MeV] at $m_q=0$.

is clear from (2), since the simulation is carried out for a single flavor. We observe a very good linear increase of m_0 as m_q decreases towards $m_q=0$, at least down to $m_q \approx 0.025$ ($K=0.168$), which corresponds to $m_\pi/m_\rho=0.42$. If we evaluate the value of m_0 at the vanishing quark mass by a linear extrapolation, we find $m_0=0.518(25)$ or $m_0=751(39)$ MeV in physical units using $a^{-1}=1.45(3)$ GeV estimated from the ρ meson mass data of Ref. [7] extrapolated to K_c. Let us emphasize that an increase of m_0 towards the chiral limit is in marked contrast to other hadron masses, which always decrease as $m_q \to 0$, e.g., $m_\pi^2 \propto m_q$, revealing the unusual nature of the U(1) problem.

The value we obtained may be compared with the empirically estimated mass difference, $m_0^2=m_{\eta'}^2+m_\eta^2-2m_K^2 \approx (852 \text{ MeV})^2$, using the Witten-Veneziano relation [2], or a more naive value,

$$m_0^2=m_{\eta'}^2-(4m_K^2+3m_\pi^2+m_\eta^2)/8 \approx (866 \text{ MeV})^2,$$

ignoring a small mixing between η and η'. While a precise agreement is not achieved, the result is quite encouraging. We may conclude that the η' mass is understood within quenched QCD at least approximately; dynamical quarks would play a relatively minor role in producing a large η'-octet mass splitting.

Another comparison is to directly test the $U(1)$ Ward identity relation derived by Witten and Veneziano,

$$m_0^2=2N_f\chi/f_\pi^2, \tag{5}$$

with χ the topological susceptibility in the pure gauge theory. We calculate χ by applying the cooling method [8] to our 300 pure gauge configurations on a $12^3\times 20$ lattice with 25 cooling sweeps. The result is $\chi=4.76(36)\times 10^{-4}$. Using $f_\pi(K=K_c)=0.0676(24)$ obtained from the same gauge configurations with the aid of the improved perturbation correction [9], we obtain $m_0=1146(67)$ MeV with the use of (5) for $N_f=3$. This value seems higher than our result and also than the empirical value. The disagreement might be ascribed to the chiral symmetry breaking of the Wilson quark action which introduces extra terms in the U(1) Ward identity.

We also note that the authors of Ref. [4] obtained $m_0=920$ MeV [10] at $m_\pi/m_\rho=0.71$ and 570 MeV at $m_\pi/m_\rho=0.34$ with $a^{-1}=1.81$ GeV from an exponential fit to the point-to-point η' propagator using 10 configurations generated with a nonstandard gauge action on an $8^3\times 16$ lattice.

Let us remark on the obvious origins of systematic errors in our result: (i) Our calculation is made at $\beta=5.7$ with a relatively large lattice spacing $a \approx 0.14$ fm to keep the physical lattice size large enough while retaining the computational demand to a modest amount with a lattice of size $(12^3\times 20)$-$(16^3\times 20)$; this coupling is still not in the region where scaling of physical quantities is expected; (ii) the calculation is made only up to $K=0.168$; the possibility is not excluded that m_0 increases faster than the linear behavior in m_q close to the chiral limit; (iii) the error may also arise from the neglect of dynamical quarks.

Our final consideration concerns implications of our result on the importance of the spurious infrared divergences that possibly appear in the chiral limit in the quenched approximation [11]. These divergences originate from the double pole $m_0^2/(p^2+m_\pi^2)^2$ in the disconnected two-quark-loop amplitude of the η' propagator. The one-loop order in chiral perturbation theory, for example, the pion mass is given by [11]

$$(m_\pi^{\text{1-loop}})^2=m_\pi^2\left[1-\delta\ln\frac{m_\pi}{\Lambda}\right], \tag{6}$$

where $\delta=m_0^2/8\pi^2N_ff_\pi^2$ and Λ is a cutoff of chiral perturbation theory. For the π-π scattering amplitude [12], the one-loop diagram formed by the two double-pole propagators yields a divergent imaginary part at threshold in the s chanel, and in the t channel gives a contribution of the form

$$\delta a_0=\frac{1}{1536\pi^3}\frac{1}{f_\pi^4 m_x}\left(\frac{m_0^2}{N_f}\right)^2 \tag{7}$$

to the s-wave scattering length, which diverges as $m_\pi \to 0$. With our result for m_0 we find that the coefficient of the logarithm in (6) has a magnitude $\delta \sim 0.1$ at $m_\pi/m_\rho \sim 0.5$. Quenched pion mass data in the region $m_\pi/m_\rho \gtrsim 0.5$ do not show evidence of the logarithm with such a small coefficient [13]. It is intriguing to note, however, that a recent result [14] with Kogut-Susskind quarks at $\beta=6.0$ for much smaller quarks masses corresponding to $0.5 \lesssim m_\pi/m_\rho \gtrsim 0.3$ show a deviation from a linear behavior $m_\pi^2 \propto m_q$ and can be fitted with (6) with $\delta \sim 0.15$, a value roughly consistent with that we obtained for $K=0.168$ ($m_\pi/m_\rho \sim 0.4$). For the scattering length (7) we expect a much smaller correction $\delta a_0/a_0^{I=0} \sim 0.003$ at $m_\pi/m_\rho \sim 0.5$ where a_0 is the current algebra prediction for the $I=0$ scattering length $a_0^{I=0}=7m_\pi/32\pi f_\pi^2$.

In conclusion, we have shown that the mass difference between η' and pseudoscalar octet mesons can be calculated with the use of the variant wall source method proposed in our previous publication, albeit within the quenched approximation. The estimated mass difference increases as m_q decreases, and its value extrapolated to $m_q=0$ is close to that determined experimentally, indicating that the long-standing U(1) problem would be solved in lattice QCD and that its essential part is understood within the quenched approximation.

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