Energy Spectrum of Homogeneous and Isotropic Turbulence in Far Dissipation Range

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The Kolmogorov relation for the third order structure function is used to derive the energy spectrum in the far dissipation range ($k \to \infty$). With no unspecified constants and methods from matched asymptotic expansions, a uniformly valid form for the inertial through the dissipative ranges is obtained. An analogous energy spectrum is presented. This is compared with the results of physical and numerical experiments on the energy spectra $E(k)$. The theoretical predictions are found to deviate by not more than a few percent from data in the entire range of wave numbers where the energy spectrum $E(k)$ varies by more than 30 orders of magnitude.

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In this Letter we address the problem of the energy spectrum $E(k)$ of fully developed turbulent flow in the far dissipation range ($k \to \infty$). It is generally accepted that the full energy spectrum consists of three dynamically different ranges. At the largest scales $l \approx L$, where L is the size of the system, the dynamics are not universal and depend on the details of the flow: geometry, instability mechanisms leading to the turbulence production, external fields, etc. In the inertial range of scales $l_d \ll l \ll L$ strong nonlinearity leads to approximately universal behavior independent of both L and molecular viscosity ν, l_d being the viscous scale. This is the range in which scaling relations hold, and velocity fluctuations can be characterized by the Kolmogorov-like spectrum. In the viscosity dominated interval $l \ll l_d$ the intensity of the velocity fluctuations rapidly decreases with the wave number k. This region inherits a form of universality from the inertial range. A rigorous mathematical estimate shows that $E(k) = O(e^{-ck})$, with the unspecified constant c [1]. Studies have suggested a prefactor of k^3 but give no first principles determination of the exponential constant [2–4]. Our result completely specifies all constants and relates the prefactor to the inertial range. In addition, the inertial and far dissipative ranges emerge from a single universal spectrum.

To set the stage for the analysis of this range, consider a flow stirred by a force $\mathbf{f}$ having nonzero Fourier components $\mathbf{f}(k)$ in the interval $0 < k \approx k_0 \approx 1/L$. In the case of statistically steady flow the following relation can be derived [5]:

$$S_3 = -\frac{4}{5}\epsilon x + \frac{6}{x^4}\int_0^x y^4 \overline{\Delta u \Delta f}\, dy + 6\nu \frac{\partial S_2}{\partial x}, \qquad (1)$$

where $S_n = \overline{(\Delta u)^n}$, $\Delta u = u(\mathbf{X}+x) - u(\mathbf{X})$, and $\Delta f = f(\mathbf{X}+x) - f(\mathbf{X})$. Here u and f represent the x components of the velocity and force fields, respectively, and x is the displacement in the x direction.

At scales $l \ll L$ this relation simplifies to [6–8],

$$S_3 = \overline{[u(\mathbf{X}) - u(\mathbf{X}+x)]^3} = -\frac{4}{5}\epsilon x + 6\nu \frac{dS_2}{dx}. \qquad (2)$$

Expressions (1) and (2) are consequences of the Navier-Stokes equations for an incompressible fluid where the dissipation rate ϵ is defined as $\epsilon = \frac{1}{2}\nu\overline{(u_{i,j}+u_{j,i})^2}\left(= 15\nu\overline{(\partial u/\partial x)^2}\right)$.

From the smoothness of derivatives of the Navier-Stokes equations it can be reasonably argued that S_n is smooth, if not analytic, at the origin. Thus $S_n = O(x^n)$ for small x and hence (2) yields

$$S_2 = \frac{e}{15\nu}x^2 + O(x^4), \qquad (3)$$

where parity considerations dictate the error estimate.

The energy spectrum can be directly related to the second order structure function by observing that

$$S_2 = \overline{[u(\mathbf{X}) - u(\mathbf{X}+x)]^2} = 2\left(u_0^2 - \overline{u(\mathbf{X})u(\mathbf{X}+x)}\right) = 2[U(0) - U(x)], \qquad (4)$$

where $u_0^2/2$ is the mean *energy* and $U(x)$ defines the two-point velocity correlation function, which is seen to be even in x and vanishes as $|x| \uparrow \infty$. Then as is well known the three-dimensional energy spectrum E is expressible in terms of the one-dimensional spectrum by

$$E(k) = k^3 \frac{\partial}{\partial k}\left(\frac{1}{k}\frac{\partial U}{\partial k}\right), \qquad (5)$$

where

$$U(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}U(x)dx. \qquad (6)$$

A Fourier transformation using only the expansion (3) leaves us with delta functions and derivatives of it in wave-number space. Correctly interpreted, this states that the principal support of $E(k)$ is at the origin to within the approximations which have been used. The

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limit, $k \to \infty$, in which we are interested is indistinguishable from zero in this approximation. Physically what is missing is the nonlinear interaction between modes which is the only source supplying energy to the small scales where it is dissipated by viscous mechanisms. Without this term there is no way to balance the viscosity. Thus, the problem of the dissipation range spectrum is a strong coupling problem, not accessible to simple perturbation expansions in powers of the nonlinearity. This was realized a long time ago by Kraichnan [2], Orszag [3], and Kuzmin and Patashinskii [4] who developed the low-order renormalized perturbation expansion to attack this problem.

We now use relation (2) to extend the expansion so as to include terms which arise from nonlinear interactions. For this purpose we recall that $\overline{(\partial u/\partial x)^3} = S\overline{(\partial u/\partial x)^2}^{3/2}$ defines the parameter of skewness, S. It then follows that in the limit $x \to 0$

$$S_3 = -S\left(\frac{\epsilon}{15\nu}\right)^{3/2} x^3 + O(x^5), \tag{7}$$

and from this that

$$6\nu S_2 = -\frac{S}{4}\left(\frac{\epsilon}{15\nu}\right)^{3/2} x^4 + \frac{2}{5}\epsilon x^2 + O(x^6). \tag{8}$$

In deriving Eq. (8) the $O(x^3)$ contribution coming from the second term on the right side of (1) has been neglected as small. It is easy to show that when $k_0 x \ll 1$ this contribution is $O(k_0^3 x^3)$. We illustrate this point in an example of a flow stirred by a white, in-time, random Gaussian force defined by the correlation function

$$\overline{f_i(k) f_j(k')} = \epsilon \frac{\delta(k^2 - k_0^2)}{k} \delta(k + k')\delta(t - t').$$

In physical space this yields the exact relation that

$$\overline{\Delta u \Delta f} = \frac{2}{3}\epsilon(1 - \cos k_0 x).$$

If this is substituted into (1) then

$$\frac{1}{x^4}\int_0^x y^4 \overline{\Delta u \Delta f}\, dy \approx \frac{1}{21}\epsilon x O(k_0^2 x^2),$$

which is small compared to $\overline{(\Delta u)^3} \approx S(\epsilon/15\nu)^{\frac{3}{2}} x^3$ when $k_0 \to 0$ for $\nu = O(1)$ and $\epsilon = O(1)$.

To normalize the above results we fix the viscous scale to be the Kolmogorov length $l_d = \eta = (\nu^3/\epsilon)^{1/4}$ and the integral scale by $L = u_0^3/\epsilon$. We can then write

$$F(\tilde{x}) = \frac{15 S_2}{\epsilon^{2/3}\eta^{2/3}} = \frac{15 S_2}{\epsilon^{1/2}\nu^{1/2}} = \tilde{x}^2 - \frac{1}{a^2}\tilde{x}^4 + O(\tilde{x}^6), \tag{9}$$

where $\tilde{x} = x/\eta$ and $a^2 = 8(15)^{3/2}/5S$. In a more general case when k_0 is finite it follows from the relations derived above that

$$\frac{1}{a^2} = \frac{5S}{8} 15^{-\frac{3}{2}} + \frac{15}{84}(k_0\eta)^2. \tag{10}$$

The relation (10), derived for the case of turbulent flow stirred by white-in-time random force, is important for our understanding of finite Reynolds number flows, when $k_0\eta$ is not extremely small. It follows from (10) that for a typical value of $S \approx 0.5$, the ratio $k_0\eta = O(\mathrm{Re}^{-\frac{3}{4}})$ in the Kolmogorov turbulence must be small for the parameter a to reach its asymptotic value. It will be clear below that a determines the form of the exponential in the dissipation range energy spectrum. Thus, even small finite Reynolds number corrections to the value of a are important in the limit $k \to \infty$.

In the above normalization the correlation function is given by

$$C(\tilde{x}) = U/u_0^2 = 1 - \beta^2 F(\tilde{x}) = 1 - \beta^2\left(\tilde{x}^2 - \frac{\tilde{x}^4}{a^2}\right) + O(\tilde{x}^6), \tag{11}$$

with $\beta = (\eta/L)^{1/3}/(30)^{1/2}$. The normalized energy spectrum is given by

$$E = \tilde{k}^3 \frac{\partial}{\partial \tilde{k}}\left(\frac{1}{\tilde{k}}\frac{\partial}{\partial \tilde{k}} C(\tilde{k})\right), \tag{12}$$

where

$$C(\tilde{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tilde{k}\tilde{x}} C(\tilde{x}) d\tilde{x} \tag{13}$$

and F now refers to the normalized spectrum.

The singularity structure of $C(\tilde{x})$ in the complex $\tilde{x}$ plane determines the behavior of $C(\tilde{k})$ for $|\tilde{k}| \uparrow \infty$, since the evaluation of (13) can proceed by distorting the integration path in the complex $\tilde{x}$ plane. We observe that the series for $C(\tilde{x})$ (11) appears as an alternating sum. Although only three terms are present in (11), one may demonstrate that this is rigorously true. From this we may also prove that the circle of convergence is limited by a singularity on the negative real axis of x^2. This suggests that the complex $\tilde{x}^2$ plane be mapped by the Euler transformation

$$w = \frac{\tilde{x}^2}{\alpha + \tilde{x}^2}, \qquad \tilde{x}^2 = \frac{\alpha w}{1 - w},$$

where the constant $\alpha > 0$ is still to be determined. Thus, as a function of w, F has a convergent power series about the origin, given by

$$F = \frac{\alpha w}{1 - w} - \frac{1}{a^2}\left(\frac{\alpha w}{1 - w}\right)^2 + O(w^3) = \alpha w + \alpha w^2 - \frac{\alpha^2 w^2}{a^2} + O(w^3). \tag{14}$$

If we take $\alpha = \alpha^2/a^2$, i.e., $a^2 = \alpha$, then $F = \alpha w + O(w^3)$ so that the second order term in (14) is eliminated from the series.

Thus, it follows that

$$F = \frac{a^2\tilde{x}^2}{a^2 + \tilde{x}^2} + O\left(\left(\frac{\tilde{x}^2}{a^2 + \tilde{x}^2}\right)^3\right), \tag{15}$$

a form which suggests poles at $\tilde{x} = \pm ia$. In this form the expansion has the appearance of a development in Padé approximants.

Next we reason that for $\tilde{x} = O(a)$ the function $F(\tilde{x})$ is well defined for $\tilde{x}$ which is real. For $\tilde{x}$ large we are physically in the inertial range and Kolmogorov's argument, used here as an assumption, yields

$$\frac{S_2}{\epsilon^{2/3}\eta^{2/3}} \sim K\tilde{x}^{2/3},$$

where K is the Kolmogorov constant. (Our deliberations extend to more general power laws and we discuss this later.) Arguments based on matched asymptotics suggest that the above expansion, say (14), can be regarded as the inner expansion of

$$F = \frac{\tilde{x}^2}{\left[1 + \frac{3}{2a^2}\tilde{x}^2\right]^{2/3}}. \qquad (16)$$

To the present order this yields

$$C(\tilde{x}) = 1 - \beta^2 \frac{\tilde{x}^2}{\left[1 + \frac{3}{2a^2}\tilde{x}^2\right]^{2/3}}. \qquad (17)$$

The Fourier transform of this expression also gives rise to distributions at the origin. This, however, is a consequence of the neglect of the integral scales in (17). In order to consider the spectrum for $|k| \uparrow \infty$ and to restrict attention to the inertial and dissipative ranges it is appropriate to neglect distributions at the origin and we write

$$C(\tilde{k}) \sim \beta^2 \frac{\partial^2}{\partial k^2} \frac{1}{2\pi} \int \frac{e^{-i\tilde{k}\tilde{x}}\, d\tilde{x}}{\left[1 + \frac{3}{2a^2}\tilde{x}^2\right]^{2/3}}$$
$$= A[\kappa^{1/6}K_{11/6}(\kappa) + \kappa^{5/6}K_{5/6}(\kappa)], \qquad (18)$$

where

$$A = \frac{2^{4/3}a^3\beta^2}{3^{3/2}\sqrt{\pi}\Gamma(2/3)}. \qquad (19)$$

$\kappa = \sqrt{\frac{2}{3}}ak\eta$ and K_ν is the modified Bessel function of order ν.

On applying (12) to (18) we find

$$\frac{E(\kappa)}{A} = 5\kappa^{7/6}K_{\frac{17}{6}}(\kappa) + \kappa^{\frac{13}{6}}K_{\frac{23}{6}}(\kappa), \qquad (20)$$

where (20) is valid for large κ. The energy spectrum as given by (20) supplies a uniformly valid form covering the two universal ranges, viz., the inertial and far dissipative ranges.

In the limit of relatively small κ the expression (20) gives the Kolmogorov spectrum. When $\kappa \to \infty$,

$$\frac{E(\kappa)}{A} \sim (\kappa^{\frac{5}{3}} + \gamma\kappa^{\frac{2}{3}})e^{-\kappa}, \qquad (21)$$

with $\gamma = \frac{39}{8} + \frac{1}{2}(\frac{23}{6})^2$.

Comparisons of the predictions based on (20) with experimental data are presented in Figs. 1–4. Figure 1 shows the experimental data on the one-dimensional energy spectrum $E_1(k/k_p)$, obtained for different turbulent flows (compilation due to She [9]). The characteristic wave number k_p corresponds to the maximum of the function $k^2E(k)$. The experimental data were fitted [9] using the function $E_e(k) = (1.14k^{-\frac{5}{3}} + 0.626k^{-1})\exp(-0.57k)$. The ratio of the theoretically derived spectrum (20) to the experimental fit, $E_e(k)$ shown in Fig. 1, is shown in Fig. 2. As is seen, the theoretical prediction given by (20) deviates from the fitting function in the entire interval $10^{-4} < k/k_p < 10$ by not more than a few percent. The largest deviation ($\approx 10\%$) occurs in the dissipation range, $k/k_p \approx 10$, where the experimental scatter is rather large.

In Fig. 3 the energy spectrum obtained in numerical simulations by Chen *et al.* [10] is shown. This low Reynolds number flow is remarkable due to the unusually large dissipation range, accurately resolved in this state-of-the-art numerical experiment (256^3 Fourier modes). The measured energy spectrum varied from $E(k) \approx 1$ at the low wave-number part of the spectrum to $E(k) \approx 10^{-40}$ at the ultraviolet cutoff $k \approx 100$, thus covering

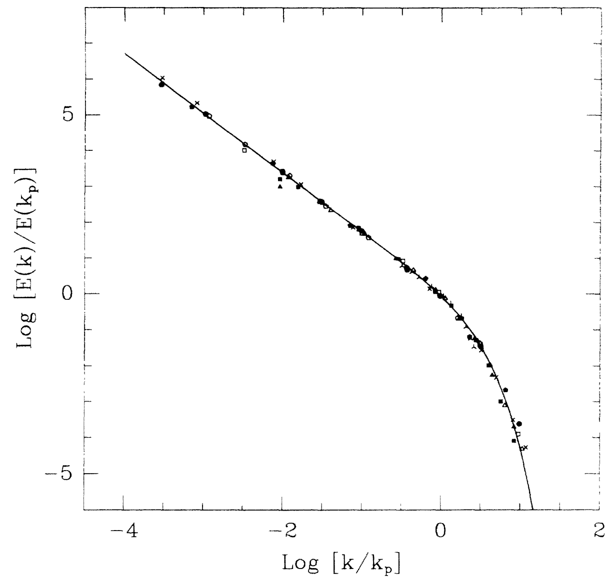

FIG. 1. Fit of experimental data (as compiled in Ref. [9]).

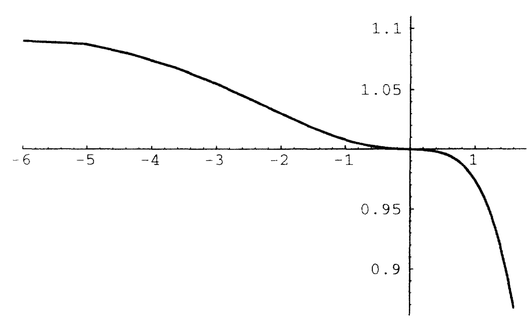

FIG. 2. Ratio of (20) to experimental fit used in Fig. 1 [ratio vs $\log(k/k_p)$].

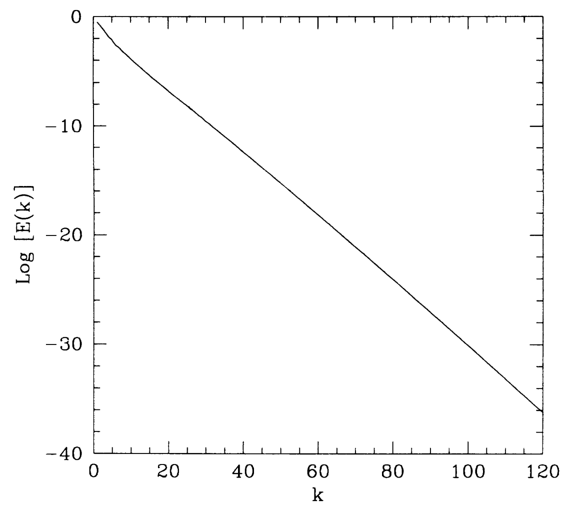

FIG. 3. Numerical spectrum from Ref. [10].

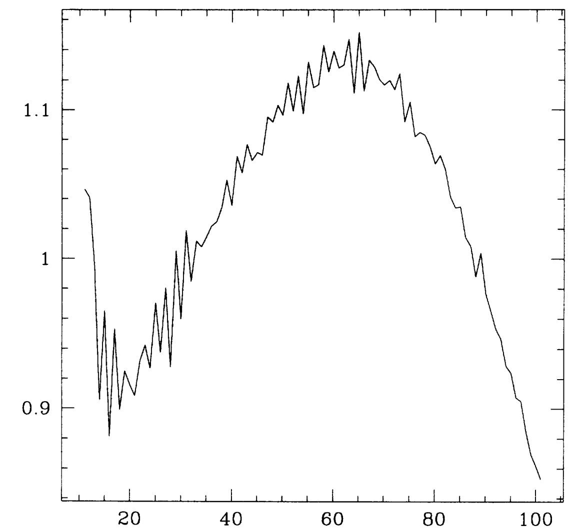

FIG. 4. Ratio of (20) to numerical spectrum shown in Fig. 3 (ratio vs k).

approximately 40 orders of magnitude of $E(k)$ variation. The flow was stirred in the small wave-number interval $1 < k < 3$. Since the Kolmogorov scale in this experiment is $k_d \approx 8.7$, parameter $k_0\eta$ is not small and has to be taken into account in the evaluation of the constant a in (10). Substituting $k_0 = 2$ into (10) leads to $a \approx 7$. The ratio of the theoretical prediction (20) with $a = 6.1$ and the results of the numerical simulation are presented in Fig. 4. As we see, the relative error does not exceed more than 10% in the entire interval $10 < k < 100$, where the energy spectrum varies by ≈ 35 orders of magnitude.

The relation (16) is based on the assumption that the inertial range is characterized by the Kolmogorov spectrum. However, if a more general power law $S_2 \propto x^\nu$ is assumed then the above analysis gives, instead of (16), $F = x^2/[1 + \frac{2}{2-\nu}(\frac{x}{a})^2]^{\frac{2-\nu}{2}}$. In this case also, all formulas can be evaluated in terms of the corresponding modified Bessel functions. In view of the close agreement that we have found using $\nu = 2/3$ we have not deemed it necessary to search for a non-Kolmogorov value of ν to better fit the data. (Only the exponents in the prefactor are modified in this case.)

The results derived in this work underline the importance of finite Reynolds number effects in the determination of the scaling exponents. Indeed, in the not-too-large Reynolds number flows, the slope of $E(k)$ in the inertial range is sensitive to the magnitudes of the derivative skewness S, cutoffs k_d and k_0, and parameter a given by expression (10). It is easy to see that the value of the exponents, obtained without accurate analysis of corrections to scaling coming from the Bessel functions in (20), can be erroneous. These effects might be especially crucial in the evaluation of such fine features like intermittency exponents in the high-order moments $S_n(x)$. There, the magnitudes of the corresponding cutoffs (k_{dn}, k_{0n}), high-order derivative skewnesses S^n, and other parameters of the scaling functions can vary with order n, thus leading to apparent incorrect scaling exponents. Only by using more sophisticated means of data processing, which include scaling functions, can one accurately determine the scaling exponents of the higher-order moments of structure functions. This resembles the situation in the theory of critical phenomena: Only after taking into account correction to scaling does an unambiguous determination of the critical exponents become possible.

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