Universal Scaling Laws in Fully Developed Turbulence

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The inertial-range scaling laws of fully developed turbulence are described in terms of scalings of a sequence of moment ratios of the energy dissipation field ϵ_ℓ coarse grained at inertial-range scale ℓ. These moment ratios $\epsilon_\ell^{(p)} = \langle\epsilon_\ell^{p+1}\rangle/\langle\epsilon_\ell^p\rangle$ $(p = 0,1,2,...,)$ form a hierarchy of structures. The most singular structures $\epsilon_\ell^{(\infty)}$ are assumed to be filaments, and it is argued that $\epsilon_l^\infty \sim l^{-2/3}$. Furthermore, a universal relation between scalings of successive structures is postulated, which leads to a prediction of the entire set of the scaling exponents: $\langle{\epsilon_\ell}^p\rangle \sim \ell^{\tau_p}$, $\tau_p = -\frac{2}{3}p + 2[1-(\frac{2}{3})^p]$ and $\langle\delta v_\ell^p\rangle \sim \ell^{\zeta_p}$, $\zeta_p = \frac{p}{9} + 2[1-(\frac{2}{3})^{p/3}]$.

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An intriguing aspect of fully developed turbulence is the possible existence of universal scaling behavior of small-scale fluctuations. In large Reynolds number flows, at spatial scales ℓ called inertial-range scales ($\ell_0 \gg \ell \gg \eta$, where ℓ_0 is the energy-injection scale and η is the molecular dissipation scale), turbulent fluctuations reach a statistically quasiequilibrium state which is characterized by a continuous energy flux from large to small scales. The universal scaling behavior refers to the observation that all moments of fluctuations at scale ℓ have a power-law dependence on ℓ, and the scaling exponents are universal. Two fluctuating quantities are of special interest: the energy dissipation averaged over a ball of size ℓ, ϵ_ℓ, and the velocity differences across a distance ℓ, δv_ℓ. The scaling behavior of ϵ_ℓ and δv_ℓ is expressed as

$$\langle\delta v_\ell^p\rangle \sim \ell^{\zeta_p}, \quad \langle\epsilon_\ell^p\rangle \sim \ell^{\tau_p}. \qquad (1)$$

Under the Kolmogorov refined similarity hypothesis [1], which has recently received both experimental and numerical support [2], ζ_p and τ_p are related, for positive p, by

$$\zeta_p = p/3 + \tau_{p/3}. \qquad (2)$$

The existence of such a universal state was first postulated by Kolmogorov [3] in 1941, who then presented a theory from which the scaling exponents are predicted with no adjustable constant. Kolmogorov (K41) assumes that all statistically averaged quantities at scale ℓ depend only on the mean dissipation rate $\bar{\epsilon}$ and ℓ. Dimensional considerations then lead to the predictions that $\tau_p = 0$ and $\zeta_p = p/3$, because $\langle\epsilon_\ell^p\rangle \sim \bar{\epsilon}^p$ is independent of ℓ. At $p = 2$, it results in the famous 2/3 law for the averaged kinetic energy fluctuations $\langle\delta v_\ell^2\rangle$. Nevertheless, extensive experimental [4] and numerical [5] studies during the last 30 years provide evidence that K41 is not accurate: ζ_p seems to deviate significantly from $p/3$ for $p > 3$. This has been referred to as intermittency corrections to K41. A number of phenomenological models [1,6–12] have since been proposed to address this correction. All existing models, however, appeal to adjustable parameters, in the determination of ζ_p and τ_p, which either have no physical meaning, or cannot be determined by plausible physical arguments.

In this Letter, we present a set of hypotheses about the statistical structures of small-scale fluctuations in turbulence. These assumptions lead to predictions of the entire set of scaling exponents of dissipation moments and velocity structure functions, without appealing to adjustable parameters. Here, we characterize the energy dissipation field ϵ_ℓ [13] by a hierarchy of fluctuation structures: $\epsilon_\ell^{(p)}$ $(p = 0,1,2,...,)$ defined by the ratio of successive dissipation moments $\langle\epsilon_\ell^{p+1}\rangle/\langle\epsilon_\ell^p\rangle$. The two extreme characteristic structures of the hierarchy, $\epsilon_\ell^{(0)}$ and $\epsilon_\ell^{(\infty)}$, are associated, respectively, to the mean fluctuation structure $\bar{\epsilon}$ and to filamentary fluid structures [14–16], in accordance to the two-fluid model [17]. As $\ell \to 0$, the mean dissipation $\epsilon_\ell^{(0)}$ is kept constant, while $\epsilon_\ell^{(\infty)}$ shows divergent scale dependence. A simple argument leads to the determination of the scaling behavior of $\epsilon_\ell^{(\infty)}$: $\epsilon_\ell^{(\infty)} \sim \ell^{-2/3}$. It is further assumed that the scaling behavior of any intermediate fluctuation structures, say $\epsilon_\ell^{(p+1)}$, is determined by a universal relation involving only $\epsilon_\ell^{(p)}$ and $\epsilon_\ell^{(\infty)}$. We then obtain an expression for the scaling exponents $\tau_p = -2p/3 + 2[1-(2/3)^p]$.

Specifically, the intensity of the pth order dissipation structures $\epsilon_\ell^{(p)}$ is given by

$$\epsilon_\ell^{(p)} = \frac{\langle\epsilon_\ell^{p+1}\rangle}{\langle\epsilon_\ell^p\rangle} = \frac{\int \epsilon_\ell^{p+1} P(\epsilon_\ell)d\epsilon_\ell}{\int \epsilon_\ell^p P(\epsilon_\ell)d\epsilon_\ell} = \int \epsilon_\ell Q_p(\epsilon_\ell)d\epsilon_\ell, \quad (3)$$

where $P(\epsilon_\ell)$ is the probability density function (PDF) of ϵ_ℓ, and $Q_p(\epsilon_\ell) = \epsilon_\ell^p P(\epsilon_\ell)/\int \epsilon_\ell^p P(\epsilon_\ell)d\epsilon_\ell$. Note that $Q_p(\epsilon_\ell)$ is a weighted PDF for which $\epsilon_\ell^{(p)}$ is the mathematical expectation. For reasonable $P(\epsilon_\ell)$ (with a single maximum and exponentially decaying tail), $\epsilon_\ell^{(p)}$ is a monotonically increasing sequence of p. Also, for large p, $Q_p(\epsilon_\ell)$ is strongly peaked around $\epsilon_\ell^{(p)}$; indeed, it can be shown that the variance of $Q_p(\epsilon_\ell)$ tends to zero as p goes to infinity, if $\epsilon_\ell^{(\infty)} < \infty$. Therefore, $Q_p(\epsilon_\ell)$ for large p

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captures fluid structures with specific dissipation intensity $\epsilon_\ell^{(p)}$. Moreover, as $p \to \infty$, $\epsilon_\ell^{(p)}$ naturally approaches $\epsilon_\ell^{(\infty)}$.

Working with the quantities $\epsilon_\ell^{(p)}$ is motivated as follows. It has been observed in numerical experiment [18] that dissipation of high intensities displays a higher degree of coherence in space, and can therefore have different scaling properties. Since $\epsilon_\ell^{(p)}$ probe dissipation structures of increasing intensities as p increases, the variation of their scaling with p gives directly a differentiation of physical properties of these structures. We now proceed in determining the scaling properties of $\epsilon_\ell^{(p)}$. First, by definition, $\epsilon_\ell^{(0)}$ is scale independent. It is just the mean field that Kolmogorov [3] used for determining all scaling properties of dissipation and velocity structure functions. The presence of anomalous scaling laws stems from a divergent scaling dependence of $\epsilon_\ell^{(\infty)}$ as $\ell \to 0$ due to the presence of intermittent structures. $\epsilon_\ell^{(\infty)}$ is the coarse-grained intensity of the most intermittent structures which, we assume, are isolated filaments. They are geometrical objects of dimension one embedded in three-dimensional space. The scaling behavior of $\epsilon_\ell^{(\infty)}$ is determined by the following argument. Dimensionally, it can be estimated by a kinetic energy divided by a time: $\epsilon_\ell^{(\infty)} \sim \delta E^\infty / t_\ell$. The anomalous scaling can enter here either through δE^∞ or through t_ℓ. We chose to have no anomalous scaling in t_ℓ, which amounts to setting a uniform time scale for the dissipation of various intensities. This yields $t_\ell \sim \bar{\epsilon}^{-1/3}\ell^{2/3}$. The underlying physical picture is, roughly speaking, that the mixing rate of fluid elements which determine this time scale for the dissipation is homogeneous in space, irrespective of the amplitude of dissipation intensity (see the next paragraph). The largest available kinetic energy to be dissipated is δv_0^2, which corresponds to the situation of quasidiscontinuity in velocity fluctuations. We then obtain

$$\epsilon_\ell^{(\infty)} \sim \bar{\epsilon}\left(\frac{\ell}{\ell_0}\right)^{-2/3} \sim \ell^{-2/3}. \qquad (4)$$

A fluid mechanical picture of those extremely intermittent events $\epsilon_\ell^{(\infty)}$ is as follows. Associated with a strong filamentary structure are quasi-two-dimensional spiral motions around the filament axis with only a small velocity component along the axis [19]. Such motions represent a remarkable coherence with respect to lower intensity fluctuations. These filamentary structures constantly entrain surrounding, less ordered fluid elements which, after being absorbed into the structure, also display quasi-two-dimensional motions. This entraining process therefore effectively dissipates their kinetic energy fluctuation along the axis of the filament, and since surrounding fluids do not move in any preferred direction, the dissipated energy is of the order of their total (fluctuating) kinetic energy ($\sim \delta v_0^2$). The assumptions in the last paragraph imply that in the most coherent state, absorbed (dissipated) kinetic energy by coarse-grained structures is independent of the length scale ℓ and attains the typical amplitude of the large-scale kinetic energy. On the other hand, the rate of entrainment (the time scale of the dissipation) is independent of the degree of the coherence of the structures; this rate is associated with motions of filamentary structures as a whole. This assumption needs experimental verifications.

This picture points out the nature of intermittency growth. It is the tendency towards the formation of local coherent structures that drives a strong deviation from the mean fluctuation level. In regions of these coherent structures, an equipartition of kinetic energy is realized (velocity difference is independent of length scale). Recall that moving towards equipartition is the statistical interpretation of 3D energy cascade. While most turbulent structures cannot reach such an equipartition state because of the disordered motions of smaller eddies, the most intermittent structures, which correspond to rare and exceptional trajectories in phase space, have effectively achieved this goal by establishing remarkable coherence locally.

It follows from (3) and (4) that as $p \to \infty$, $\tau_{p+1} - \tau_p \to -2/3$, or $\tau_p \to -\frac{2}{3}p + C_0$. This asymptotical law has received some support from the recent published experimental measurements [20]. A geometrical interpretation of the constant C_0 can be made through the Legendre transform: C_0 is the codimension of the very high intensity structures as $\ell \to \eta$ (see more discussion below). Since these high intensity structures are filaments, the Hausdorff dimension is 1; thus $C_0 = 2$. We then obtain

$$\tau_p = -\tfrac{2}{3}p + 2 + o(p) \quad (p \to \infty). \qquad (5)$$

Characteristic dissipation intensities $\epsilon_\ell^{(p)}$ for finite p correspond to less singular and less coherent structures which may be geometrically more ribbon or sheetlike [15]. The scaling of $\epsilon_\ell^{(p)}$ characterizes how singular the pth order structures are. These structures are results of the dynamics which intend to form very singular ($\epsilon_\ell^{(\infty)}$) structures, but fail to fully accomplish it because of abundant surrounding disorder. In physical space, pth order structures of intensity $\epsilon_\ell^{(p)}$ are mostly likely to be surrounded by structures of $(p-1)$th order having slightly lower intensity. In time, structures of intensity $\epsilon_\ell^{(p-1)}$ are also likely to occur before the formation of even higher intensity structures of pth order. This reasoning suggests that the scaling behavior of $\epsilon_\ell^{(p)}$ may be related to that of $\epsilon_\ell^{(p-1)}$ and $\epsilon_\ell^{(\infty)}$. We propose that the scaling behavior of $\epsilon_\ell^{(p+1)}$ is determined by a *universal* relation involving $\epsilon_\ell^{(p)}$ and $\epsilon_\ell^{(\infty)}$ for all p. Specifically, we postulate

$$\epsilon_\ell^{(p+1)} = A_p {\epsilon_\ell^{(p)}}^{\beta} {\epsilon_\ell^{(\infty)}}^{1-\beta}, \quad 0 < \beta < 1, \qquad (6)$$

so that as $p \to \infty$, $\epsilon_\ell^{(p)} \to \epsilon_\ell^{(\infty)}$. A_p are constants which are independent of ℓ, but may depend on the specific geometry of the flow system. (There is no reason that they are universal.) The correctness of (6) needs experimental verification. We believe that this relation is a consequence of some hidden (statistical) symmetries in the solution of the Navier-Stokes equations.

It follows from (3) and (6) that for each $p = 0, 1, 2, ...,$

$$\tau_{p+2} - (1+\beta)\tau_{p+1} + \beta\tau_p + \tfrac{2}{3}(1-\beta) = 0. \quad (7)$$

It can be verified that the asymptotic solution (5) is satisfied exactly. Let us pose $\tau_p = -2/3p + 2 + f(p)$ with $f(\infty) = 0$; we then obtain a second order homogeneous difference equation for $f(p)$:

$$f(p+2) - (1+\beta)f(p+1) + \beta f(p) = 0. \quad (8)$$

The only nontrivial solution of (8) is $f(p) = \alpha\beta^p$. The whole solution τ_p must satisfy two boundary conditions $\tau_0 = 0$ and $\tau_1 = 0$. The first condition is derived from an assumption that the support of dissipation is finite in the limit of zero viscosity, and the second is exact. These two conditions uniquely determine α and β: $\alpha = -C_0 = -2$ and $\beta = 2/3$. The final expression of τ_p is then

$$\tau_p = -\tfrac{2}{3}p + 2[1 - (\tfrac{2}{3})^p]. \quad (9)$$

Using (2), we predict that

$$\zeta_p = p/9 + 2[1 - (\tfrac{2}{3})^{p/3}]. \quad (10)$$

The present model predicts an energy spectrum of turbulence

$$E(k) \sim k^{-\frac{29}{9}+2(\frac{2}{3})^{2/3}} \approx k^{-\frac{5}{3}-0.03}. \quad (11)$$

A widely used quantity [4,5], μ, characterizing intermittency correction to K41 is the deviation from K41 of the exponent for the sixth order velocity structure function: $\zeta_6 = 2 - \mu$. Experimental measurements [4,5] give a μ between 0.2 and 0.25. Here, we predict $\mu = 2/9$.

There has been much effort in both experimental studies of laboratory turbulent flows and direct numerical simulations of the Navier-Stokes turbulence to measure the scaling exponents ζ_p of high-order velocity structure functions [4,5]. Although the accuracy of the measurements is not all satisfactory, expecially for large p where systematic errors may result from undersampling, qualitative features are clear: strong deviations from K41 occur for $p > 3$. Recently, Benzi *et al.* [21] made significant progress in accurately determining these scaling exponents. With the concept of extended self-similarity, they are able to extend by almost a decade the inertial range, allowing for significant improvement of the accuracy of the numerical values of ζ_p (under a few percent for p up to 8, private communication from Benzi). In Table I, we compare our predictions (10) to their results. It can be seen that the agreement is very good.

Legendre transform of τ_p, or ζ_p, gives a conjugate description of the multiscaling behavior of turbulence. This transformation can be calculated analytically here, which yields a function $D(h)$, usually referred to as the fractal dimension of scaling exponent h. For ζ_p, the fractal dimension $D(h)$ is

$$D(h) = 1 + c_1(h - \tfrac{1}{9}) - c_2(h - \tfrac{1}{9})\ln(h - \tfrac{1}{9}), \quad (12)$$

$$c_1 = 3\left(\frac{1 + \ln\ln\frac{3}{2}}{\ln\frac{3}{2}} - 1\right), \quad c_2 = \frac{3}{\ln\frac{3}{2}}.$$

It is interesting to note that $D(1/3) \approx 2.97$, $D(0.381) = 3$, and $h_{\min} = 1/9 > 0$. The function $D(h)$ was the basis for the multifractal interpretation [6] of turbulence scalings. It is natural to ask what their interpretation is in the present framework. We would like to argue that each individual exponent h in (12) may not be dynamically relevant; instead, a mean exponent $\bar{h}(p)$ should be introduced. Observe that

$$\tau_{p+1} - \tau_p = \int_p^{p+1} \frac{\partial\tau_p}{\partial p}dp = \int_p^{p+1} h(p)dp = \bar{h}(p). \quad (13)$$

Here, $h(p)$ are exponents associated with the dissipation field. It is now clear that the scaling exponents of $\epsilon_\ell^{(p)}$, $\tau_{p+1} - \tau_p$, are just the mean exponents $\bar{h}(p)$ integrated from p to $p+1$. Recall that $\epsilon_\ell^{(p)}$ is the expectation of $Q_p(\epsilon_\ell)$ which are the central objects in the present description. The assumption (6) can also be understood as an invariant relation between the mean and relative dispersion of $Q_p(\epsilon_\ell)$ with respect to scale transformation. Since $Q_p(\epsilon_\ell)$ has finite dispersion at small p, they mostly select geometrical objects with a range of dissipation intensities around $\epsilon_\ell^{(p)}$, reflecting uncertainty about tracing dynamically revelant objects of low intensities. This is quite consistent with the observation [18] that low intensity fluctuations have a considerable degree of randomness. $\bar{h}(p)$ can be interpreted as the mean scaling exponent of this range of objects. The uncertainty in exponents naturally leads to uncertainty in dimension estimate. As p increases, however, both exponents and dimensions become properties more specific to fluctuation intensities at $\epsilon_\ell^{(p)}$, since the variance $\mathrm{Var}_{Q_\infty}(\epsilon_\ell)$ tends to zero. Only there can geometrical objects in space be clearly identified. These are the asymptotic filamentary structures.

TABLE I. Comparison of the measured scaling exponents ζ_p from experiments [21] and the theoretical prediction (10).

Order p	Experiment [21] ζ_p	Theory (10) ζ_p	K41 $\zeta_p = p/3$
2	0.71	0.696	0.667
3	1	1	1
4	1.28	1.279	1.333
5	1.53	1.538	1.667
6	1.78	1.778	2
7	2.01	2.001	2.333
8	2.22	2.211	2.667
10	2.60	2.593	3.333

In 1974, Kraichnan [22] pointed out that once K41 is abandoned, a Pandora's box of possibilities is open and specific contact with the Navier-Stokes dynamics must be made in order to determine scaling properties of the inertial range. Phenomenological models, both the family of models of log-normal type for ϵ_ℓ [1,9–11] and those based on multiplicative processes [7,8] (motivated by the idea of multifractality [6]), are unable to relate themselves to the Navier-Stokes fluid dynamics. Consequently, parameters in those models cannot be determined by plausible arguments. The present work attempts to make a concrete connection to the Navier-Stokes dynamics. To summarize, two major assumptions are made in the present description: First, the self-similar structure expressed by (6) relates the scaling properties of structures of various intensities, which is, we believe, due to the quadratic nonlinearity of the underlying dynamics. Indeed, similar arguments can be carried out to analyze Burgers equation, a strongly intermittent case. Second, there is a class of structures that represents the most intermittent events; they are asymptotically filaments. The nature of these asymptotic flow structures is a specific property of the three-dimensional incompressible flows: only filamentary structures seem to be mechanically stable. In two-dimensional incompressible flows with strong enstrophy cascades, the corresponding structures may be instead point vortices, and the quantitative predictions should therefore be a little different. What is interesting is that these two ingredients seem to fully specify the scaling properties of turbulence at inertial-range scales. Much work is in progress in digesting the mathematical content of the present work; results will be communicated in the near future.

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[1] A.N. Kolmogorov, J. Fluid Mech. **13**, 82–85 (1962).

[2] G. Stolovitzky, P. Kailasnath, and K.R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992); S.T. Thoroddsen and C.W. Van Atta, Phys. Fluids A **4**, 2592 (1992); S. Chen, G.D. Doolen, R.H. Kraichnan, and Z.-S. She, Phys. Fluids A **5**(2), 458–463 (1993).

[3] A.N. Kolmogorov, C.R. Acad. Sci. URSS **30**, 301–305 (1941).

[4] There are many results, see, e.g., F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Antonia, J. Fluid Mech. **140**, 63–89 (1984).

[5] A representative work is A. Vincent and M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).

[6] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 71.

[7] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521–3531 (1984).

[8] C. Meneveau and K.R. Sreenivasan, Phys. Rev. Lett. **59**, 1424–1427 (1987).

[9] B. Castaing, J. Phys. (Paris) **50**, 147–156 (1989).

[10] L.C. Andrews, R.L. Phillips, B.K. Shivamoggi, J.K. Beck, and M.L. Joshi, Phys. Fluids A **1**, 999 (1989).

[11] S. Kida, J. Phys. Soc. Jpn. **60**, 5 (1990).

[12] Z.-S. She and S.A. Orszag, Phys. Rev. Lett. **66**, 1701 (1991).

[13] The dissipation we consider here can be either a 1D cut $\nu(du/dx)^2$ or the sum of all components $\nu s_{ij}s_{ij}$. We believe that both display similar behavior at high Reynolds numbers.

[14] E.D. Siggia, J. Fluid Mech. **107**, 375–406 (1981).

[15] Z.-S. She, E. Jackson, and S.A. Orszag, Nature (London) **344**, 226–228 (1990).

[16] S. Douady, Y. Couder, and M.-E. Brachet, Phys. Rev. Lett. **67**, 983 (1991).

[17] Z.-S. She, Fluid Dyn. Res. **8**, 143–158 (1991).

[18] Z.-S. She, E. Jackson, and S.A. Orszag, Proc. R. Soc. London A **434**, 101 (1991).

[19] At the completion of this work, we noticed that a similar suggestion was made in a recent work by I. Procaccia and P. Constantin, Phys. Rev. Lett. **70**, 3416–3419 (1993).

[20] A. Bershadskii and A. Tsinober, Phys. Rev. E **48**, 282 (1993).

[21] R. Benzi, S. Ciliberto, C. Baudet, F. Massaioli, R. Tripiccione, and S. Succi, Phy. Rev. E **48**, 29 (1993).

[22] R.H. Kraichnan, J. Fluid Mech. **62**, 305–330 (1974).