

## Non-Abelian Excitations of the Quark-Gluon Plasma

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We present new, non-Abelian solutions to the equations of motion which describe the collective excitations of a quark-gluon plasma at high temperature. These solutions correspond to spatially uniform color oscillations.

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The long wavelength excitations of a quark-gluon plasma are collective excitations which are described by nonlinear equations generalizing the classical Yang-Mills equations in the vacuum. Most studies of such equations have been limited so far to their weak field limit, where the modes reduce essentially to Abelian-like plasma waves. The purpose of this Letter is to present new, truly non-Abelian solutions that we have obtained recently. At leading order in the gauge coupling  $g$  (we assume  $g \ll 1$  in the high temperature deconfined plasma), the collective dynamics is entirely described by a set of effective equations for the soft gauge mean fields  $A_a^\mu(x)$  which describe the long wavelength ( $\lambda \sim 1/gT$ ) and low frequency ( $\omega \sim gT$ ) excitations ( $T$  denotes the temperature) [1,2]. [Throughout this work, the greek indices refer to Minkowski space, while the latin subscripts are color indices for the adjoint representation of the gauge group  $SU(N)$ .] The equations satisfied by  $A_a^\mu(x)$  are

$$[D^\nu, F_{\nu\mu}(x)]_a = j_{\mu a}^{\text{ind}}(x), \quad (1)$$

where  $D^\mu = \partial^\mu + igA^\mu(x)$ ,  $A^\mu \equiv A_a^\mu T^a$ , and  $F^{\mu\nu} = [D^\mu, D^\nu]/ig = F_a^{\mu\nu} T^a$ . The induced current  $j_\mu^{\text{ind}}$  describes the response of the plasma to the color gauge fields  $A_a^\mu$ . It is proportional to the fluctuations in the phase-space color densities of quarks and gluons. Its expression is [1,3]

$$j_{\mu a}^{\text{ind}}(x) = 3\omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu W_a(x;v) \quad (2)$$

for retarded boundary conditions [ $A_a^\mu(x) \rightarrow 0$  as  $x_0 \rightarrow -\infty$ ]. The notations here are as follows:  $\omega_p^2 \equiv (2N + N_f)g^2 T^2/18$  is the *plasma frequency*,  $v^\mu \equiv (1, \mathbf{v})$ , where  $\mathbf{v} \equiv \mathbf{k}/k$  is the velocity of the hard particle with momentum  $\mathbf{k}$  ( $k \equiv |\mathbf{k}|$ ), and the integral  $\int d\Omega$  runs over all the directions of the unit vector  $\mathbf{v}$ . Furthermore, the functions  $W_a(x;v)$  are generally nonlocal and nonlinear functionals of the gauge fields, defined by

$$W_a(x;v) = \int_0^\infty du U_{ab}(x, x-vu) \mathbf{v} \cdot \mathbf{E}_b(x-vu), \quad (3)$$

where  $E^i \equiv F^{i0}$  is the chromoelectric field, and  $U(x,y)$  is the parallel transporter along the straight line  $\gamma$  joining  $x$  and  $y$ . The induced current (2) is covariantly conserved,  $[D^\mu, j_\mu^{\text{ind}}(x)] = 0$ . The energy density of an arbitrary gauge field configuration in the plasma has been recently computed as [3]

$$T^{00}(x) = \frac{1}{2} [\mathbf{E}_a(x) \cdot \mathbf{E}_a(x) + \mathbf{B}_a(x) \cdot \mathbf{B}_a(x)] + \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} W_a^0(x;v) W_a^0(x;v), \quad (4)$$

with  $B_a^i(x) \equiv -\frac{1}{2} \epsilon^{ijk} F_a^{jk}(x)$ . Note that  $T^{00}(x)$  is manifestly positive, and so is therefore the excitation energy  $\mathcal{E}(t) = \int d^3x T^{00}(t, \mathbf{x})$ , at any time  $t$ .

The solutions to the non-Abelian field equations (1) have direct physical relevance: They correspond to collective excitations of the high temperature quark-gluon plasma. In this Letter, we study particular solutions which are uniform in space. (Another interesting limit, that of a static field configuration, has been considered recently [4], with the conclusion that no finite energy solution exists.) For convenience, we choose the temporal gauge,  $A_a^0(x) = 0$ . Hence,  $A_a^\mu(x) \equiv (0, \mathbf{A}_a(t))$ .

For uniform fields, the functions  $W_a(t;v)$  are simple local functionals of the gauge potentials [3],

$$W_a(t;v) = -\mathbf{v} \cdot \mathbf{A}_a(t), \quad (5)$$

and the same holds for the induced current (2),  $j_a^{\text{ind}}(t) = -\omega_p^2 \mathbf{A}_a(t)$ ,  $\rho_a^{\text{ind}}(t) = 0$ , and for the energy density (4),

$$T^{00}(t) = \frac{1}{2} \left[ \frac{d\mathbf{A}_a}{dt} \cdot \frac{d\mathbf{A}_a}{dt} + \omega_p^2 \mathbf{A}_a \cdot \mathbf{A}_a \right] + \frac{g^2}{4} f^{abc} f^{ade} (\mathbf{A}_b \cdot \mathbf{A}_d) (\mathbf{A}_c \cdot \mathbf{A}_e). \quad (6)$$

We have used here the expressions of the field strengths in terms of the vector potentials,

$$\mathbf{E}_a(t) = -\frac{d\mathbf{A}_a}{dt}, \quad \mathbf{B}_a(t) = \frac{g}{2} f^{abc} \mathbf{A}_b(t) \times \mathbf{A}_c(t). \quad (7)$$

The field equations (1) reduce then to the following equations for the vector potentials  $A^i(t) = A_a^i(t) T^a$ :

$$\frac{d^2 A^i}{dt^2} + \omega_p^2 A^i + g^2 ([A^i, A^j], A^j) = 0, \quad (8)$$

together with a constraint (Gauss's law)

$$\left[ A^i, \frac{dA^i}{dt} \right] = 0, \quad (9)$$

corresponding to the component  $\mu = 0$  of Eq. (1). These equations are similar to the classical Yang-Mills equa-

tions in the vacuum, which have been extensively investigated already for the case of SU(2) [5-8]. They differ, however, by the presence of the thermal mass term  $\omega_p^2 A^i$ , which, as we shall see, has a strong effect on the dynamics. Note incidentally that, for gauge fields satisfying Gauss's law (9), the Poynting vector  $S^i \equiv T^{i0}$  vanishes [3].

The constraint (9) is satisfied, in particular, by field configurations of the form  $A_a^i(t) = \mathcal{A}_a^i h^i(t)$  (no summation over  $i$ ), with constant  $\mathcal{A}_a^i$  and arbitrary functions  $h^i(t)$ . Indeed, for such fields, the three color vectors  $\{A_a^i(t)\}$  and  $\{dA_a^i/dt\}$  ( $i=1,2,3$ ) are parallel in color space. In the rest of this Letter we restrict ourselves to SU(2), and assume  $\mathcal{A}_a^i = \delta_a^i$ . This ansatz is equivalent to that proposed by Basyan, Matinyan, and Savvidy [5], up to a trivial global gauge rotation [8]. The functions  $h_i(t)$  then satisfy

$$\frac{d^2 h_1}{dt^2} + \omega_p^2 h_1 + g^2 h_1 (h_2^2 + h_3^2) = 0, \tag{10}$$

plus two similar equations for  $h_2$  and  $h_3$ . The associated energy density,

$$T^{00} = \frac{1}{2} \sum_i \left[ \left( \frac{dh_i}{dt} \right)^2 + \omega_p^2 h_i^2 \right] + \frac{g^2}{2} (h_1^2 h_2^2 + h_1^2 h_3^2 + h_2^2 h_3^2), \tag{11}$$

is an integral of motion and acts as an effective Hamiltonian for the functions  $h_i(t)$ .

At this point, it is convenient to make a scale transformation and define the dimensionless variable  $x \equiv \omega_p t$ , as well as the dimensionless functions  $f_i(x) \equiv (g/\omega_p) h_i(t)$ . We also assume  $f_3 = 0$ , with no significant loss of generality. We obtain then for  $f_1$  and  $f_2$  the coupled nonlinear equations

$$\begin{aligned} \ddot{f}_1(x) + \{1 + [f_2(x)]^2\} f_1(x) &= 0, \\ \ddot{f}_2(x) + \{1 + [f_1(x)]^2\} f_2(x) &= 0, \end{aligned} \tag{12}$$

where the overdots indicate derivatives with respect to  $x$ . The energy density (11) is then

$$T^{00} = \frac{\omega_p^4}{g^2} \frac{1}{2} (\dot{f}_1^2 + \dot{f}_2^2 + f_1^2 + f_2^2 + f_1^2 f_2^2) \equiv \frac{\omega_p^4}{g^2} \mathcal{H}. \tag{13}$$

The Hamiltonian  $\mathcal{H}$  is that of a system of two nonlinearly coupled harmonic oscillators with coordinates  $f_1$  and  $f_2$ . Note that the different parameters characterizing the initial system (i.e.,  $\omega_p, g, T^{00}$ ) combine into a single, dimensionless one,  $\theta^2 \equiv (g^2/\omega_p^4) T^{00}$ , which measures the total energy of the mechanical system:  $\mathcal{H} = \theta^2$ .

In deriving Eq. (1), we have assumed the gauge fields to be weak ( $A \lesssim T$ ) and slowly varying ( $\partial A \sim gT A$ ) [1]. Remembering that  $\omega_p \sim gT$ , one sees that these conditions imply  $|f_i(x)| \lesssim 1$  and  $|\dot{f}_i(x)| \sim |f_i(x)|$ , so that  $\theta$  is, at most, of order unity,  $\theta \lesssim 1$ . These limitations are consistent with the dynamics described by Eqs. (12). The quadratic terms  $f_1^2 + f_2^2$  in the Hamiltonian (13), which

originates from the thermal mass, play an essential role in this respect. Because of them, and of energy conservation, a trajectory  $\{f_i(x)\}$  cannot leave the bounded domain delineated by the equipotential lines  $f_1^2 + f_2^2 + f_1^2 f_2^2 = 2\theta^2$  (Fig. 1). Let us assume  $\theta \lesssim 1$ . Then, since  $|f_i(x)| \leq \sqrt{2}\theta$  (see Fig. 1), it follows that  $|f_i(x)| \lesssim 1$  for any  $x$ . Furthermore, the quantity  $[1 + f_i^2(x)]$ , which plays the role of an effective frequency squared for the motion in the direction  $j \neq i$ , remains of order 1 for any  $x$ , so that  $f_i$  and  $\dot{f}_i$  remain of the same order of magnitude. Consequently, if the conditions on the functions  $f_i$  mentioned above are satisfied for some  $x_0$ , then they are valid for any  $x$ . The situation here is different from the classical, vacuum case, where the equipotential lines are given by  $h_1^2 h_2^2 = (2/g^2) T^{00}$ ; then, for given energy  $T^{00}$ , the motion could extend arbitrarily far along the  $h_1$  or the  $h_2$  axis. Using the language of dynamical systems, one may observe that, for the system (12), the origin of the four-dimensional phase space  $(f_1, \dot{f}_1, f_2, \dot{f}_2)$  is an *elliptic fixed point*, corresponding to a neutrally stable equilibrium [9]. At  $T=0$ , this same point is marginally unstable. Thus, apart from ensuring bounded trajectories  $\{|f_i(x)| \lesssim 1\}$ , which, as we have discussed, guarantees the consistency of our equations, the quadratic terms  $f_i^2$  in Eq. (13) also improve the stability of the system.

Let us consider now simple, periodic solutions to the system (12).

(a) The simplest motion is one dimensional along the axis 1 or 2. Assume, e.g.,  $f_2 = 0$ . We get then a simple harmonic oscillator equation for  $f_1$ ,

$$f_1(x) + f_1(x) = 0. \tag{14}$$

The general solution is  $f_1 = a_1 \cos x + a_2 \sin x$ , with  $a_1^2 + a_2^2 = 2\theta^2$ . The corresponding field configuration,  $A^1(t) = (C_1 \cos \omega_p t + C_2 \sin \omega_p t) T^1$  [with  $C_i \equiv (\omega_p/g) a_i$ ], and  $A^2 = A^3 = 0$ , describes periodic oscillations (with period

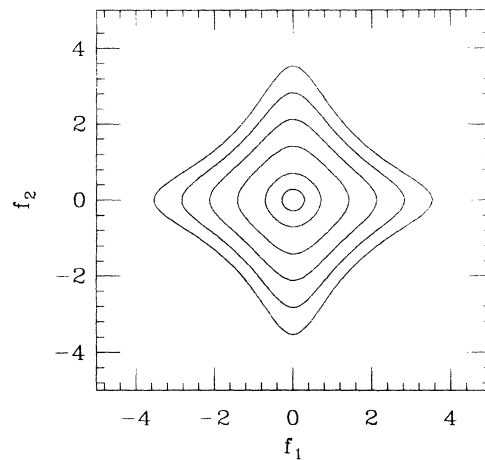


FIG. 1. Equipotential curves for the Hamiltonian  $\mathcal{H}$ , Eq. (13). The parameter  $\theta$  ranges from 0.2 (inner curve) to 2.5 (outer curve).

$T_0 = 2\pi/\omega_p$ ) in space-color direction 1.

(b) Another one-dimensional, but less trivial example corresponds to periodic solutions with  $f_2(x) = \pm f_1(x) \equiv \pm f(x)$ . These solutions describe in or out of phase oscillations of the colors 1 and 2. The function  $f(x)$  satisfies the nonlinear equation

$$\ddot{f}(x) + f(x) + f^3(x) = 0, \quad (15)$$

for which the integral of motion is

$$\theta^2 = \dot{f}^2 + f^2 + \frac{1}{2}f^4. \quad (16)$$

A particular solution to (15) is

$$f(x) = f_\theta \text{cn}[(2\theta^2 + 1)^{1/4}(x - x_0); k], \quad (17)$$

where  $\text{cn}(x; k)$  is the Jacobi elliptic cosine of argument  $x$  and modulus  $k$ , and  $x_0$  is the arbitrary origin of the time. The parameters  $k$  and  $f_\theta$  are related to  $\theta$  by

$$k = \frac{1}{\sqrt{2}} \left[ 1 - \frac{1}{\sqrt{2\theta^2 + 1}} \right]^{1/2}, \quad (18)$$

and

$$f_\theta = (\sqrt{2\theta^2 + 1} - 1)^{1/2}. \quad (19)$$

It can be easily seen that  $f_1 = \pm f_2 = \pm f_\theta$  are the coordinates of the intersection points between the trajectories and the equipotential lines in Fig. 1. The solution (17) corresponds to the initial conditions  $f(x_0) = f_\theta$  and  $\dot{f}(x_0) = 0$ . For the in-phase oscillations, the associated gauge potentials are  $A^1(t) = h(t)T^1$ ,  $A^2(t) = h(t)T^2$ , and  $A^3(t) = 0$ , where  $h(t) = (\omega_p/g)f(\omega_p t)$  is a periodic function, with period

$$\mathcal{T} = \frac{4}{\omega_p} \frac{1}{(2\theta^2 + 1)^{1/4}} K(k), \quad (20)$$

and  $K(k)$  is the complete elliptic integral of modulus  $k$ . Since  $\theta \lesssim 1$ ,  $\mathcal{T}$  remains of order  $\mathcal{T}_0 = 2\pi/\omega_p$ . Thus, it is the plasma frequency  $\omega_p \sim gT$  which controls the time variation of the nonlinear color oscillations (17), for any value of the parameter  $\theta \lesssim 1$ . This is true, in particular, for *small oscillations*, that is, when the energy density  $T^{00} \rightarrow 0$ , so that  $\theta \approx f_\theta \ll 1$  and  $k \rightarrow 0$ . Then, the periodic solution (17) reduces to a simple harmonic oscillation, and  $\mathcal{T} \rightarrow \mathcal{T}_0$ . In contrast, in the zero temperature case, as  $T^{00} \rightarrow 0$ , the corresponding period is diverging [5]. [The  $T=0$  expression for  $\mathcal{T}$  is obtained from Eq. (20) by replacing  $(2\theta^2 + 1)^{1/4} \omega_p \rightarrow (2g^2 T^{00})^{1/4}$  and  $k \rightarrow 1/\sqrt{2}$ .]

For this solution, the nontrivial field strengths are  $E_a^i = -\delta_a^i \dot{h}$ , with  $i=1,2$ , and  $B_a^3 = gh^2 \delta_a^3$ . Accordingly, the three vectors  $\mathbf{E}_1$ ,  $\mathbf{E}_2$ , and  $\mathbf{B}_3$  are mutually orthogonal.

An important question concerns the stability of the periodic orbits of the system (12). For the corresponding equations at zero temperature, it has been shown, through a numerical analysis, that unstable periodic trajectories exist [5]. In particular, the in-phase oscillations of two colors [the vacuum analog of (17)] are unstable

[6,7]. In the case of the system (12), the presence of the linear terms  $f_1$  and  $f_2$ , which originate from the thermal mass in (11), guarantees the stability of the solutions, for small oscillations and for most initial conditions; indeed, when  $|f_i| \ll 1$ ,  $f_1^2 + f_2^2 \gg f_1^2 f_2^2$  and the dynamics is essentially that of harmonic oscillators. This is verified in the numerical analysis of the system (12) performed in Ref. [10] in the physically different context of a classical Yang-Mills-Higgs system in the vacuum. There, the mass terms for the gauge fields are generated through spontaneous symmetry breaking and are proportional to the vacuum expectation value of the scalar field,  $\eta = \langle \phi \rangle$ . The correspondence between the two sets of equations is simply given by  $\omega_p^2 \rightarrow g^2 \eta^2/2$ . The numerical experiment in Ref. [10] clearly shows that for  $\theta \ll 1$  (e.g., for  $\theta \approx 0.2$ ) most of the trajectories are quasiperiodic. As  $\theta$  increases, the stochastic motion strongly develops and, beyond a critical value  $\theta_c^2 \approx 6.6$ , it fills the entire permissible range of motion in the phase space. As the above value of  $\theta_c$  stays in the limits of validity of the present approach, it would be worthwhile to further investigate the physical significance of this transition. Analogous findings, concerning the nonintegrability and the stochastic behavior of the dynamical system (12), are presented in Refs. [11] and [12].

In conclusion, we have studied here global color oscillations of the high temperature quark-gluon plasma, by looking at the spatially uniform limit of the effective equations of motion derived in Ref. [11]. We have shown that the plasma frequency  $\omega_p = gT(N + N_f/2)^{1/2}/3$  characterizes the inertial properties of the plasma both for the Abelian-like and for the genuine non-Abelian collective motion. The phase-space motion is quasiperiodic for small oscillations and for most initial conditions. We have obtained explicit nonlinear solutions for SU(2) which describe in or out of phase oscillations of two colors [the generalization to the in phase oscillations of the three colors, i.e.,  $h_1 = h_2 = h_3$  in Eq. (10), is straightforward [3]]. Such solutions can be embedded in any larger SU( $N$ ) theory in the standard way [13]. As a final remark let us emphasize that we have assumed here a weak coupling regime in which the damping of the color modes can be ignored. A recent discussion of issues related to this question can be found in Refs. [14,15].

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