Light Deflection in Perturbed Friedmann Universes

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A new formula for light deflection is derived using only physically observable concepts. The general result is specialized to cosmological perturbation theory and expressed in terms of gauge-invariant perturbation variables. The resulting scalar, vector, and tensor equations are supplemented by simple examples for illustration. The gravity-wave example may be of more than academic interest and even represent an alternative way to detect gravitational waves.

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The propagation of photons in a perturbed Friedmann universe leads to perturbations in the cosmic microwave background which have been detected recently, and to perturbations of light from distant sources like quasars. Several formulas to calculate the perturbation of photon energies (Sachs-Wolfe effect) have been derived so far [1]. In this Letter, we investigate in general the deflection of light in the framework of cosmological perturbation theory. A formula equivalent to our result (12) for scalar perturbations is derived in [2], where the influence of gravitational lensing on small angle fluctuations of the cosmic microwave background is discussed [3].

On large scales the geometry of our Universe can be approximated to a precision of about 10^{-5} by a homogeneous and isotropic Friedmann-Lemaître spacetime. Deviations from homogeneity and isotropy may thus be treated in first order perturbation theory. Cosmological perturbation theory was first investigated by Lifshitz [4]. Since only the perturbed geometry is physically observable, the choice of a background universe on which perturbations are defined is somewhat arbitrary and is called a choice of gauge. Only quantities whose background contributions vanish are gauge independent [5]. Physically measurable quantities can always be expressed in terms of such gauge-invariant variables, which usually have a simple geometrical meaning [6].

We therefore derive a formula for light deflection in terms of gauge-invariant variables. Throughout, we choose the metric signature $(-,+,+,+)$.

On their way from the last scattering surface into our antennas, microwave photons travel through a perturbed Friedmann geometry. Thus, even if the photon temperature was completely uniform at the last scattering surface, we receive it slightly perturbed. In addition, photons traveling through a perturbed universe are deflected. We now calculate this deflection in first order perturbation theory.

Metrics which are conformally equivalent, $d\tilde{s}^2 = a^2 ds^2$, have the same lightlike geodesics; only the corresponding affine parameters are different. Since Friedmann-Lemaître models are conformally flat, we may thus discuss the propagation of light in a perturbed Minkowski geometry. This simplifies things greatly. We denote the affine parameter by λ and the tangent vector to the geodesic by $n = dx/d\lambda$, $n^2 = 0$, with unperturbed values $n^0 = 1$ and $\boldsymbol{n}^2 = 1$. The tangent vector of the perturbed geodesic is given by $(1, \boldsymbol{n}) + \delta n$. The geodesic equation for the metric

$$ds^2 = (\eta_{\alpha\nu} + h_{\alpha\nu})dx^\alpha dx^\nu$$

yields to first order in $h_{\mu\nu}$

$$\delta n^\mu|_i^f = -\eta^{\mu\nu}[h_{\nu 0} + h_{\nu j}n^j]_i^f + \frac{\eta^{\mu\sigma}}{2}\int_i^f h_{\rho\nu,\sigma}n^\rho n^\nu d\lambda \ , \quad (1)$$

where the integral can be performed along the unperturbed photon trajectory from some initial spacetime point p_i to the end point p_f. On the right hand side, unperturbed values for n^μ can be inserted. Starting from this general relation, one obtains the Sachs-Wolfe effect by discussing the perturbation of $n \cdot u$, where u is the four-velocity of an observer [1,5,7].

The direction of a light ray with respect to an observer moving with four-velocity u is given by the direction of the spacelike vector

$$n_{(3)} = [n + (u \cdot n)u]/|u \cdot n| \ , \quad (2)$$

which lives on the subspace of tangent space normal to u. We now want to define the deflection of a light ray. At the point of emission, the photon direction as measured by an observer moving according to the velocity field u is given by $n_{(3)}(p_i)$. Correspondingly, at the point of detection it is given by $n_{(3)}(p_f)$. We have to compare these two vectors at different points. We do not want to switch off the perturbation, since this is not a gauge-invariant concept. (There exist different averaging procedures, e.g., over different spacelike hypersurfaces, which lead to slightly different Friedmann backgrounds. Actually the difference in the obtained backgrounds can be of the same order of magnitude as the amplitude of the perturbations [5].) We thus let the observer transport their frame of reference from the point of emission to the point of detection and compare the direction of $n_{(3)}(p_f)$ with respect to the transported frame.

We do not want to restrict the observer to move on a geodesic. The correct way to transport a direction along

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a nongeodesic curve is Fermi transport (see, e.g., [8]). The Fermi transport equation is given by

$$\nabla_u n^{||}_{(3)} = (n^{||}_{(3)} \nabla_u u)u \ . \tag{3}$$

Here, we denote by $n^{||}_{(3)}$ the transported direction of emission which coincides with $n_{(3)}$ at the point of emission. This equation is uniquely specified by the requirement that Fermi transport should conserve scalar products with u and scalar products of vectors normal to u. Therefore, angles and lengths in the subspace of tangent space normal to u are conserved.

If an observer Fermi transports their frame of reference with respect to which angles are measured, they consider a light ray as not being deflected if $n^{||}_{(3)}(p_f)$ is parallel to $n_{(3)}(p_f)$. The difference between the direction of these two vectors is thus the deflection,

$$\phi e = \left[n_{(3)} - (n^{||}_{(3)} \cdot n_{(3)})n^{||}_{(3)}\right](p_f) \ . \tag{4}$$

Here e is a spacelike unit vector normal to u and normal to $n^{||}_{(3)}$ which determines the direction of deflection and ϕ is the deflection angle. [Note that (4) is a general formula for light deflection in an arbitrary gravitational field. Up to this point we did not make any assumptions about the strength of the field.] Clearly, due to this definition, the deflection angle will in general depend on the path which the observer chooses to move from p_i to p_f. But we shall see in the following that, at least in cases where the gravitational field originates from a spatially confined mass distribution, the observer can always move on a path far away from all masses, so that the path dependent contribution becomes negligible.

For a spherically symmetric problem, e is uniquely determined by the above orthogonality conditions since the path of light rays is confined to the plane normal to the angular momentum. In the general case, when the angular momentum of photons is not conserved, e still has 1 degree of freedom. We now calculate ϕe perturbatively. Let us define the perturbed quantities:

$$n = (1, \boldsymbol{n}) + \delta n \ \text{ with } \ \boldsymbol{n}^2 = 1 \ ,$$
$$u = (1,0) + \delta u = (1 + \tfrac{1}{2}h_{00}, \boldsymbol{v}),$$
$$n_{(3)} = (0, \boldsymbol{n}) + \delta n_{(3)},$$

and

$$n^{||}_{(3)} = (0, \boldsymbol{n}) + \delta n^{||}_{(3)} \ .$$

The perturbation δn is given in (1). From (2) we obtain

$$\delta n_{(3)} = \epsilon u + \epsilon(0, \boldsymbol{n}) + \delta n - \delta u \ ,$$
$$\epsilon = [n^i v^i - \delta n^0 + \tfrac{1}{2}h_{00} + n^i h_{i0}] \ . \tag{5}$$

The Fermi transport equation (3) yields

$$\delta(n^{||}_{(3)})^0 = n^i(h_{i0} + v_i), \tag{6}$$

$$\delta(n^{||}_{(3)})^j = -\frac{1}{2}\left[h_{lj}n^l|_i^f + \int_i^f dt(h_{j0,l} - h_{l0,j})n^l\right] \ . \tag{7}$$

Inserting (5)–(7) into (4) leads to

$$\phi e^0 = 0 \ , \quad \phi e^i = \delta^i - (\boldsymbol{\delta} \cdot \boldsymbol{n})n^i \ . \tag{8}$$

with

$$\delta_j = [\delta n_j - v_j + \tfrac{1}{2}h_{jk}n^k]|_i^f + \frac{1}{2}\int_i^f (h_{0j,k} - h_{0k,j})n^k \, dt \ . \tag{9}$$

This quantity is observable and thus gauge invariant. The last integral has to be performed along the path of the observer. For a finite mass distribution, at distance R away from all masses, $h_{i0} < M/R$; therefore $h_{0i,j} < M/R^2$. On the other hand, the length of the path is of order R, so that we find

$$\int_i^f (h_{0j,k} - h_{0k,j})n^k \, dt \le \frac{M}{Rv} \ .$$

The observer can thus always choose a path far away from the mass distribution so that this term can be neglected.

We now want to express the general formulas (8) and (9) in terms of gauge-invariant variables for scalar, vector, and tensor type perturbations. The most general form for *scalar* perturbation of the metric is given by

$$(h_{\mu\nu}) = \begin{pmatrix} 2A & 2B_{,i} \\ 2B_{,i} & 2(H_L - \frac{1}{3}\triangle H)\delta_{ij} + 2H_{,ij} \end{pmatrix} \tag{10}$$

and a *scalar* velocity field can be derived from a potential, $v_i = -v_{,i}$. One can show that the following combinations of these variables are gauge invariant [7,9,10]:

$$V = v - \dot{H} \ , \quad \Psi = A + \dot{H} - B \ , \quad \Phi = H_L - \tfrac{1}{3}H \ . \tag{11}$$

The variables Φ and Ψ are the so-called Bardeen potentials. In Newtonian approximation with Newtonian potential φ one finds $\Psi = -\Phi = \varphi$. With ansatz (10) Eq. (9) leads to

$$(\delta_j)^{(S)} = V_{,j}|_i^f + \int_i^f (\Phi - \Psi)_{,j} \, d\lambda \ . \tag{12}$$

For spherically symmetric perturbations, where e is uniquely defined, we can write this result in the form

$$\phi^{(S)} = V_{,i}e^i|_i^f + \int_i^f (\Phi - \Psi)_{,i}e^i \, d\lambda \ . \tag{13}$$

The first term here denotes a special relativistic spherical aberration. The second term represents gravitational light deflection, which, since it is conformally invariant, only depends on the Weyl part of the curvature. For scalar perturbations, the amplitude of the Weyl tensor is proportional to $\Psi - \Phi$ [5,6].

Neglecting the spherical abberation we obtain in the Newtonian limit

$$\phi e = e_{\text{out}} - e_{\text{in}} = -2\int_i^f \nabla_\perp \varphi \, d\lambda \ , \qquad (14)$$

which corresponds to Eq. (4.19) of Ref. [11], leading directly to the usual lens equation [11]. As an easy test we insert the Schwarzschild weak field approximation: $\Psi = -\Phi = -GM/r$. The unperturbed geodesic is a straight line, $x = (\lambda, \boldsymbol{n}\lambda + \boldsymbol{e}b)$, where b denotes the impact parameter of the photon. Inserting this into (13) yields Einstein's well known result, $\phi^{(S)} = 4GM/b$.

The most general ansatz for *vector* perturbations is

$$(h_{\mu\nu}) = \begin{pmatrix} 0 & 2B_i \\ 2B_i & H_{i,j} + H_{j,i} \end{pmatrix}, \quad v^i \ , \qquad (15)$$

where B_i, H_i, and v_i are divergence free vector fields.

The following combinations of these variables are gauge invariant [5,10]:

$$\Omega_i = v_i - B_i \ , \quad \sigma_i = \dot{H}_i - B_i \ . \qquad (16)$$

Inserting these definitions into (9), we obtain

$$(\delta_j)^{(V)} = \Omega_j|_i^f - \frac{1}{2}\left[\int_i^f (\sigma_{j,k} - \sigma_{k,j}) n^k dt + \int_i^f \sigma_{k,j} n^k d\lambda\right] . \qquad (17)$$

This result can be expressed in three dimensional notation as follows:

$$\phi \boldsymbol{e} = \boldsymbol{\delta} - (\boldsymbol{\delta}\cdot\boldsymbol{n})\boldsymbol{n} = -(\boldsymbol{\Omega} \wedge \boldsymbol{n}) \wedge \boldsymbol{n}|_i^f + \frac{1}{2}\int_i^f (\boldsymbol{\nabla} \wedge \boldsymbol{\sigma}) \wedge \boldsymbol{n}\, dt - \int_i^f [\boldsymbol{\nabla}(\boldsymbol{\sigma}\cdot\boldsymbol{n}) \wedge \boldsymbol{n}] \wedge \boldsymbol{n}\, d\lambda \ .$$

The first term is again a special relativistic "frame dragging" effect. The second term is the change of frame due to the gravitational field along the path of the observer. The third term gives the gravitational light deflection. (Special relativistic Thomas precession is not recovered in linear perturbation theory since it is of order v^2.)

This formula can be used to obtain in first order the light deflection in the vicinity of a rotating neutron star or a Kerr black hole. In suitable coordinates the metric of a Kerr black hole with mass M and angular momentum $M\boldsymbol{a}$ can be approximated by [8]

$$g_{00} = -(1 - 2m/r) + \mathcal{O}(r^{-3}), \qquad (18)$$
$$g_{0i} = 2\epsilon_{ijk} s^j x^k / r^3 + \mathcal{O}(r^{-3}),$$
$$g_{ij} = \delta_{ij}(1 - 2m/r) + \mathcal{O}(r^{-3}) \ , \qquad (19)$$

with $m = GM$, $\boldsymbol{s} = GM\boldsymbol{a}$, and $|\boldsymbol{a}| < GM$ ($|\boldsymbol{a}| = GM$ represents the extreme Kerr solution). Therefore,

$$\boldsymbol{\sigma} = -\boldsymbol{B} = -(2/r^3)\boldsymbol{s} \wedge \boldsymbol{r} \ . \qquad (20)$$

We consider an observer which emits a light ray with impact vector $b\boldsymbol{e}$ in direction $\boldsymbol{n}$ at infinity and detects it at infinity on the other side of the black hole. The observer moves to the point of detection in a wide circle around the black hole, so that the contribution from the path of the observer can be neglected. Light deflection is then given by

$$\boldsymbol{\phi}^V = \frac{4GM}{b^2}\{\boldsymbol{a} \wedge \boldsymbol{n} + 2[(\boldsymbol{a} \wedge \boldsymbol{n}) \cdot \boldsymbol{e}]\boldsymbol{e}\}. \qquad (21)$$

The size of the deflection angle is thus of the order $|\phi^{(V)}| \sim (a/b)|\phi^{(S)}|$ and is limited by

$$|\phi^{(V)}| \leq \frac{12GMa}{b^2} = \frac{3a}{b}|\phi^{(S)}| \ .$$

Only for extreme Kerr and very close encounters (where linear perturbation theory is no longer valid since $b \sim GM$), the vector contribution is of the same order of magnitude as the scalar term. For usual circumstances, $a \ll b$, the fact that the direction differs from $\boldsymbol{e}$ might open the possibility of actually observing the vector contribution.

Tensor perturbations of the metric are of the form

$$(h_{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & H_{ij} \end{pmatrix}, \qquad (22)$$

where H_{ij} is a traceless divergence free tensor field. The deflection angle then becomes

$$(\phi e_j)^{(T)} = -H_{jk} n^k|_i^f + \int_i^f (H_{lk,j} + \dot{H}_{kl} n_j) n^l n^k \, d\lambda \ . \qquad (23)$$

Only the gravitational effects remain. The first contribution comes from the difference of the metric before and after passage of the gravitational wave. Usually this term is negligible. The second term accumulates along the path of the photon.

It is well known since the work of Bondi *et al.* that gravitational waves do deflect light and can thus act as gravitational lenses [12]. Here, we evaluate (23) for a gravitational wave pulse for which the difference of the gravitational field before and after passage of the pulse is negligible:

$$\phi e_j = \int_i^f (H_{lk,j} + \dot{H}_{lk} n_j) n^k n^l \, d\lambda \ . \qquad (24)$$

During the crossing time of the photon, we approximate the pulse by a plane wave,

$$H_{jl} = \Re\{\epsilon_{jl} \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]\} \quad \text{with} \quad \epsilon_{jl} k^l = \epsilon_l^l = 0 \ .$$

For a photon with unperturbed trajectory $x = (\lambda, \boldsymbol{x}_0 + \lambda \boldsymbol{n})$, we find the deflection angle

$$|\phi| = \begin{cases} \epsilon_{lm} n^l n^m \dfrac{|\boldsymbol{k} - \omega \boldsymbol{n}|}{\omega - \boldsymbol{k} \cdot n} \cos[\alpha + (\boldsymbol{k} \cdot \boldsymbol{n} - \omega)t] & \text{for} \quad \boldsymbol{k} \neq \omega \boldsymbol{n}, \\ 0 & \text{for} \quad \boldsymbol{k} = \omega \boldsymbol{n} . \end{cases}$$

Setting $\boldsymbol{n} = (p/\omega)\boldsymbol{k} + q\boldsymbol{n}_\perp$ with $\boldsymbol{n}_\perp^2 = 1$ and $p^2 + q^2 = 1$, we have $\epsilon_{lm} n^l n^m = q^2 \epsilon_\perp$, where $\epsilon_\perp = \epsilon_{ij} n_\perp^i n_\perp^j$. Inserting this above, we obtain

$$\phi = \epsilon_\perp \sqrt{2}(1+p)\sqrt{1-p}\cos[\alpha + \omega(p-1)t] \ . \qquad (25)$$

The amplitude of the gravitational wave determines $\epsilon_\perp$ and p is given by the intersection angle of the photon with the gravitational wave as explained above.

This effect for a gravitational wave from two coalescing black holes would be quite remarkable: Since for this (most prominent) event $\epsilon_\perp$ can be as large as $\approx 0.1(R_s/r)$, light rays passing the black hole with impact parameter b would be deflected by the amount

$$\phi \approx 2''(10^4 R_S/b) \ .$$

Setting the source at distance d_{LS} from the coalescing black holes and at distance d_S from us, we observe a deflection angle

$$\beta = \phi d_{LS}/d_S \ .$$

The best source candidates would thus be quasars for which d_{LS}/d_S is of order unity for all coalescing black holes with, say, $z \leq 1$. In the vicinity of the black holes ($r \leq 10R_S$), linear perturbation theory is of course not applicable. But in the wide range $10^7 R_S > b > 10R_S$ for radio sources, and $10^4 R_S > b > 10R_S$ for optical sources, our calculation is valid and the result might be detectable.

A thorough investigation of the possibility of detecting gravitational waves of coalescing black holes out to cosmological distances by this effect may be worthwhile. The effects of light deflection and lensing by gravitational waves have also been considered in [13].

We have defined light deflection in a gauge-invariant, operational way. In general, the result depends on the path along which the observer transports their frame of reference from the point of emission to the point of detection. Our equations are derived in perturbed Minkowski space, but since Friedmann-Lemaître universes are conformally flat, they also apply for them. In a $K = 0$ Friedmann model our main results [(12), (17), and (23)] are directly valid in conformal time. For $K = \pm 1$ a conformal coordinate system has to be adapted.

More applications of light deflection can be found in a review paper on cosmological perturbation theory [5]. There the formulas derived in this Letter are used to discuss gravitational lensing by topological defects.

[1] R.K. Sachs and A.M. Wolfe, Astrophys. J. **147**, 73 (1967); L.F. Abott and M.B. Wise, *ibid.* **282**, L47 (1984); L.F. Abott and R.K. Schaefer, *ibid.* **308**, 546 (1986); N. Gouda, M. Sasaki, and Y. Suto, *ibid.* **321**, L1 (1987); Y. Suto, K. Gorski, R. Yuszkiewicz, and J. Silk, Nature (London) **332**, 328 (1988); R. Durrer, Phys. Rev. D **42**, 2533 (1990).

[2] K. Tomita, Phys. Rev. D **40**, 3821 (1989).

[3] A. Kashlinsky, Astrophys. J. **331**, L1 (1988).

[4] E.M. Lifshitz, Sov. Phys. JETP **10**, 116 (1946).

[5] R. Durrer, "Fundamentals in Cosmic Physics" (to be published).

[6] M. Bruni, G. Ellis, and P.K.S. Dunsby, Classical Quantum Gravity **9**, 921 (1992); M. Bruni, P.K.S. Dunsby, and G. Ellis, Astrophys. J. **395**, 34 (1992).

[7] R. Durrer, Phys. Rev. D **42**, 2533 (1990).

[8] N. Straumann, *General Relativity and Astrophysics* (Springer-Verlag, Berlin, 1985).

[9] J. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[10] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[11] P. Schneider, J. Ehlers, and E. Falco, *Gravitational Lensing* (Springer-Verlag, Berlin, 1992).

[12] H. Bondi, F.A.E. Pirani, and I. Robinson, Proc. R. Soc. London A **251**, 519 (1959); R. Penrose, in *Perspectives in Geometry and Relativity: Essays in Honor of Vàclav Hlavatý*, edited by B. Hoffmann (Indiana University Press, Bloomington, 1966); D.M. Zipoy, Phys. Rev. **142**, 825 (1966).

[13] E.V. Linder, Phys. Rev. D **34**, 1759 (1986); E.V. Linder, Astrophys. J. **328**, 77 (1988); V.B. Braginsky, N.S. Kaedashev, A.G. Polnarev, and I.D. Novikov, Nuovo Cimento Soc. Ital. Fis. **105B**, 1141 (1990); V. Faraoni, Astrophys. J. **398**, 425 (1992); V. Faraoni, in *Gravitational Lenses*, edited by R. Kayser, T. Schramm, and S. Refsdal (Springer-Verlag, Berlin, 1993); R. Fakir, Astrophys. J. **418**, 202 (1993); R. Fakir, Astrophys. J. (to be published).