Colliding Black Holes: The Close Limit

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The problem of the mutual attraction and joining of two black holes is of importance as both a source of gravitational waves and as a testbed of numerical relativity. If the holes start out close enough that they are initially surrounded by a common horizon, the problem can be viewed as a perturbation of a single black hole. We take initial data due to Misner for close black holes, apply perturbation theory, and evolve the data with the Zerilli equation. The computed gravitational radiation agrees with and extends the results of full numerical computations.

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The collision of two black holes is, in principle, one of the most efficient mechanisms for generation of gravitational waves. In view of the fact that the LIGO and VIRGO [1] detectors may be detecting events in the coming years, the theoretical determination of possible wave forms has become of great importance. The fact that data may be well below the noise level of the detectors may require pattern-matching techniques [2] which require accurate knowledge of radiation wave forms.

The problem of the gravitational radiation generated by colliding black holes is not only of great importance to gravitational wave astrophysics, it has also been one of the earliest applications of numerical general relativity. Smarr and Eppley [3,4], more than 15 years ago, computed the radiation wave forms for the axisymmetric problem of two holes, starting from rest and falling into each other in a head-on collision. The importance of this problem has motivated a recent reconsideration, both numerical and analytical, by Anninos *et al.* [5]. The numerical work is difficult, especially when the holes are initially close together. In that case the radiation is dominated by horizon processes. In addition, for initially close holes the radiation generated is relatively small and the numerical errors in its computation can be particularly troublesome. The purpose of this paper is to provide a method of computing the radiated power generated when the holes start off close together. Our method is based on perturbation theory and is considerably more economical than a full numerical simulation. It can also be viewed as a benchmark against which numerical codes can be checked.

For a full numerical computation of the problem, data used are those for the two throats of a momentarily static "wormhole," for which an analytic solution was given by Misner [6]. The initial data have a parameter μ_0 which can be adjusted so that the initial conditions correspond to different values of L/M, where L is the initial separa-

tion of the throats, and M is the mass of the spacetime.

For values of μ_0 corresponding to large and moderate starting values of L/M the motion of the "particles," not their black hole nature, is crucial to the generation of gravitational radiation. In this case the amount of radiation emitted can be understood with a quasi-Newtonian approximation that starts with the known radiation for a point mass falling into a hole [5]. For values above around $\mu_0 \approx 2$, this quasi-Newtonian approximation is in remarkably good agreement with the results of numerical relativity. For smaller values of μ_0, however, the approximation seriously overestimates the radiation. When the throats start off at small separation their "internal" structure cannot be ignored.

It is our purpose here to provide an understanding of the opposite limit, the limit of small μ_0 (and hence small L/M). Below we give a simple and attractive analytic result valid in the limit of small separation for the energy radiated. A rather complete understanding of the phenomenon is then afforded by this result along with the quasi-Newtonian estimate, and the bridge between the two given by the numerical relativity computations. In addition, our method gives predicted wave forms, and other features of the wave, that can be useful for verifying the accuracy of the numerical relativity computations.

It should be understood that our small-μ_0 result does not have the same robust connection with a simple physical picture as the quasi-Newtonian approximation has with the mutual in-fall of two holes from large distance. In the large distance limits the details of the choice of initial data to represent the individual holes is unimportant; it is their particlelike motion toward each other that dominates the radiation. For the initially close limit the Misner data are one possible choice for momentarily static initial conditions, but they are singled out by mathematical convenience, not by any claim that they represent the most natural, instantaneously stationary,

initial distortion of the participating throats.

In terms of bispherical coordinates μ, η, ϕ, the Misner initial data [6] take the form

$$ds_{\text{Misner}}^2 = a^2 \varphi_{\text{Misner}}^4 \left[d\mu^2 + d\eta^2 + \sin^2 \eta \, d\phi^2 \right] , \qquad (1)$$

where a is a constant with the dimension of length, and

$$\varphi_{\text{Misner}} = \sum_{n=-\infty}^{n=+\infty} \frac{1}{\sqrt{\cosh(\mu + 2n\mu_0) - \cos\eta}} . \qquad (2)$$

We now change from bispherical to spherical coordinates R, θ, ϕ and introduce a Schwarzschild radial coordinate r through $R = \left(r^{1/2} + \sqrt{r - 2M} \right)^2$ to arrive at a line element of the form

$$ds_{\text{Misner}}^2 = F(r, \theta; \mu_0) \left[\frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2 \right] , \qquad (3)$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$, and with

$$F = \left(1 - \frac{M}{2R} \right)^{-4} \left(1 + \frac{a}{R} \sum_{n=0}^{\infty} \frac{1}{\sinh n\mu_0 \sqrt{1 + 2a/R \coth n\mu_0 \cos\theta + a^2/R^2 \coth^2 n\mu_0}} \right)^4 . \qquad (4)$$

The square root in the summation has the form of the generating function for the Legendre functions $P_\ell(\cos\theta)$, so that F can be expressed as

$$F = \left[1 + \frac{2}{1 + M/2R} \sum_{\ell=2,4,\ldots}^{\infty} \kappa_\ell \left(\frac{M}{R} \right)^{\ell+1} P_\ell(\cos\theta) \right]^4 , \qquad (5)$$

with

$$\kappa_\ell \equiv \frac{1}{[4\Sigma_1(\mu_0)]^{\ell+1}} \sum_{n=1}^{\infty} \frac{(\coth n\mu_0)^\ell}{\sinh n\mu_0} . \qquad (6)$$

Here, and below, we use the notation

$$\Sigma_k(\mu_0) \equiv \sum_{1}^{\infty} (\sinh n\mu_0)^{-k} . \qquad (7)$$

Since the Misner geometry satisfies the initial value equations of general relativity, and is momentarily stationary, there exists a coordinate choice T, r, θ, ϕ such that the initial data generate a 4-geometry at $T = 0$ of the form $ds^2 = -dT^2 + ds_{\text{Misner}}^2$, and for which $\partial g_{\mu\nu}/\partial T = 0$ at $T = 0$. One can make a transformation such that the 4-geometry takes the form

$$ds^2 = -\left(1 - \frac{2M}{\tilde{r}} \right) dt^2 + F(r, \theta; \mu_0) \left[\frac{d\tilde{r}^2}{1 - 2M/\tilde{r}} + \tilde{r}^2 d\Omega^2 \right] + \mathcal{O}(t^2) . \qquad (8)$$

In this spacetime the initial data are the Misner geometry, and the initial perturbations are contained in the deviations of the Misner geometry from the Schwarzschild initial geometry. All of the information about those deviations is contained in F and in how it differs from its Schwarzschild form, unity. If $F - 1$ is initially small, the evolution of the spacetime can be viewed as a problem in the dynamics of Schwarzschild perturbations and therefore one is guaranteed to have small perturbations forever.

We have therefore cast Misner's initial data in Schwarzschild coordinates. We now need to explore the limit in which the two black holes are "close." When $\mu_0 < 1.36$ an apparent horizon at $R \approx 2M$ surrounds both throats. As μ_0 decreases further, the value of L/M decreases, and hence the ratio of L to the horizon radius decreases. As pointed out by Smarr the horizon, and the geometry outside it, should then be nearly spherical, and it is only the geometry outside the horizon that influences the radiation sent outward to infinity. Linearized perturbation theory should therefore give a good description of the generation of radiation (though not of the highly nonspherical geometry inside the horizon).

To give this picture a mathematical realization we note that the coefficients in (5) then have the small-μ_0 limit

$$\kappa_\ell \approx \frac{\zeta(\ell + 1)}{(4|\ln \mu_0|)^{\ell+1}} . \qquad (9)$$

In the spacetime corresponding to (5) we can therefore consider $\epsilon \equiv 1/|\ln \mu_0|$ to be an expansion parameter. If in (5) we keep only the leading term in ϵ for each ℓ we get

$$ds_{\text{Misner}}^2 \approx \left[1 + \frac{8}{1 + M/2R} \sum_{\ell=2,4,\ldots}^{\infty} \kappa_\ell \left(\frac{M}{R} \right)^{\ell+1} P_\ell(\cos\theta) \right]$$

$$\times \left[\frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2 \right] . \qquad (10)$$

We now argue that each multipole term can be treated individually by linearized perturbation theory. This is clearly true for the $\ell = 2$ case. To lowest order in ϵ, that is to order ϵ^3, the field equations contain only terms linear in the perturbation (i.e., linear in $1 - F$), and those terms are pure $\ell = 2$. Next consider, for example, expansion to order ϵ^7. This would contain $\ell = 6$ terms linearly, but would also contain nonlinear terms, e.g., from the square of the $\ell = 2$ term [with higher order corrections to (9) included]. But the nonlinear contributions can have no

$\ell = 6$ part. (To get an $\ell = 6$ term requires a cube of an $\ell = 2$ term, or a product of $\ell = 2$ and $\ell = 4$, both of which are higher order in ϵ.) This shows that the $\ell = 6$ part of the vacuum Einstein equations is linear in $1 - F$, and hence can be treated by linear perturbation theory. This argument easily generalizes to arbitrary ℓ.

Though each of the ℓ poles in (10) satisfies the linearized sourceless Einstein equations, they are not all the same order in ϵ. We take up here only the dominant term at small initial separation, the pure quadrupole $\ell = 2$ term. To treat this as a perturbation problem, we use the notation and formalism of Cunningham et al. [7] except that we will omit the tildes over variables; equations

from that paper will be cited as "CPM." It is important to note that the formulas in that paper, based on the work of Moncrief [8], are gauge invariant, so we need pay no attention to the coordinate gauge of (10).

From CPM (II-25),(II-26), we find that the only non-vanishing metric perturbation functions are $H_2 = K = g(r)p(\mu_0)$ where [with the notation of (7)]

$$g(r) \equiv \frac{M^3}{8R^3 (1 + M/2R)}, \qquad p(\mu_0) \equiv \frac{\Sigma_1(\mu_0) + \Sigma_3(\mu_0)}{[\Sigma_1(\mu_0)]^3}. \tag{11}$$

From CPM (II-27),(II-28) we next find

$$Q_1 = 2r \left(1 - \frac{2M}{r}\right)^2 \left[\frac{g}{1 - 2M/r} - \frac{1}{\sqrt{1 - 2M/r}} \frac{d}{dr} \left(\frac{rg}{\sqrt{1 - 2M/r}}\right)\right] + 6rg, \tag{12}$$

and, following CPM (II-31), we define

$$\psi \equiv \sqrt{\frac{4\pi}{5}} \frac{Q_1}{\lambda}, \tag{13}$$

where $\lambda \equiv 1 + \frac{3M}{2r}$. The function ψ then satisfies the $\ell = 2$ Zerilli equation [9]

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial r_*^2} + \left(1 - \frac{2M}{r}\right) \left\{\frac{1}{\lambda^2} \left[\frac{9M^3}{2r^5} - \frac{3M}{r^3} \left(1 - \frac{3M}{r}\right)\right] + \frac{6}{r^2\lambda}\right\} = 0, \tag{14}$$

where $r_* \equiv r + 2M \ln \left(\frac{r}{2M} - 1\right)$. [It is worth noting that the function Q_2 of CPM (II-28), calculated from our perturbations (12), explicitly solves the initial value constraint $Q_2 = 0$ as given in CPM (II-29)].

The form of ψ given by (12) and (13) is now taken as initial data along with the initial condition $\partial\psi/\partial t = 0$ at $t = 0$. The problem is greatly simplified by the fact that the only μ_0 dependence is contained in the multiplicative factor $p(\mu_0)$. Since the initial data are proportional to $p(\mu_0)$, it follows that the evolved wave form ψ is proportional to $p(\mu_0)$, and the radiated power and energy are proportional to $[p(\mu_0)]^2$. It is only necessary, therefore, to do one computation of the evolved wave form and radiated power. The μ_0 dependence is known at the outset.

The initial form of ψ [with $p(\mu_0)$ set to unity] is shown in Fig. 1. We evolved these data numerically with the Zerilli equation (14). The resulting wave form, at $r^* = 200$ (we use units in which $2M = 1$), as a function of t, is shown in Fig. 2. It clearly exhibits quasi-normal ringing and power-law tails corresponding to a quadrupolar perturbation. The values of the quasinormal frequencies [10] and power-law exponents [11] are in excellent agreement with theoretical values. From the evolved data we compute the radiated power, which is given by CPM (III-28),

$$\text{Power} = \frac{1}{348\pi} \left|\frac{\partial}{\partial t}\psi\right|^2. \tag{15}$$

If $p(\mu_0)$ is set to unity, this procedure gives a computed energy of 3.07×10^{-6}. The resulting energy, as a function of μ_0, is therefore

$$\text{Energy}/2M = 3.07 \times 10^{-6}p(\mu_0)^2. \tag{16}$$
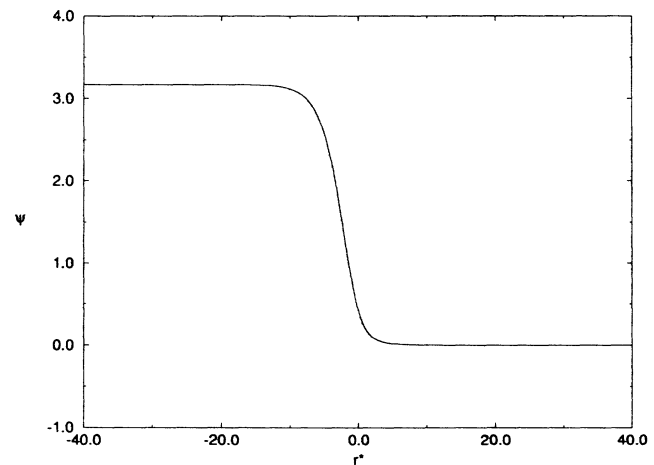
This result is displayed in Fig. 3, where it is compared

FIG. 1. The function ψ of the Cunningham-Price-Moncrief perturbation scheme for Misner's initial data. The values shown are for $p(\mu_0) = 1$, and for units in which $2M = 1$. For the r^* coordinate the horizon is at $r^* = -\infty$ and $r^* \sim r$ for large positive values.
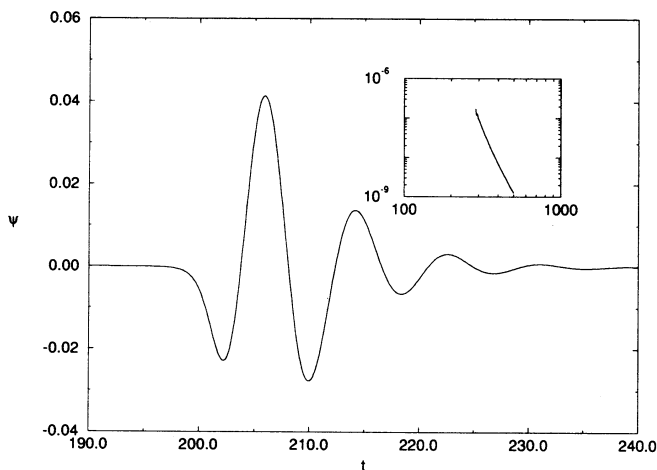
FIG. 2. Time evolution of the Misner initial data [with $p(\mu_0) = 1$, $2M = 1$], from the point of view of an observer fixed at $r^* = 200$. We see the appearance of quasinormal ringing with the predicted period of 8.4. In the inset we display in a log-log plot the late time behavior of the field, which clearly exhibits a power-law tail form with exponent -6 as predicted by theory.
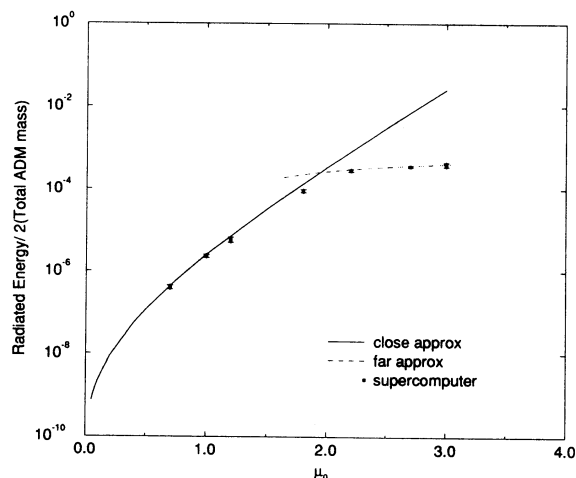

FIG. 3. The solid curve is the prediction for the radiated energy, as a function of μ_0, based on linearized perturbation theory as developed in this paper. The black dots correspond to the values of full numerical relativity and the dashed curve corresponds to a far approximation (Anninos et al. [5] and private communication).

with the numerical results reported by Anninos et al. [5]. It is intriguing that the remarkable agreement extends considerably beyond the small-μ_0 region in which our approximation is expected to be applicable.

The general method is suited to a fairly wide variety of initial data. This paper only concentrated on the aesthetically elegant Misner data as an example. The method can also be applied to initial data which are known only in numerical form and which represent perturbations of a black hole. To do this would require considerably more numerical analysis than for the Misner initial data but, when applicable, would be much quicker and less expensive than the fully numerical methods for integrating the nonlinear field equations.

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