PHYSICAL REVIEW
LETTERS

VOLUME 72 | 23 MAY 1994 | NUMBER 21

Nonlinear Interaction of Traveling Waves of Nonintegrable Equations

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(Received 22 February 1994)

We present a new methodology for deriving physically important exact solutions of certain nonintegrable equations. These solutions describe the nonlinear interaction of traveling waves. Examples include multishock and multisoliton solutions.

PACS numbers: 03.40.Kf

There exist physically significant nonlinear evolution equations, which although nonintegrable, posses exact solutions describing the interaction of two traveling wave solutions. An example is the Fitzhugh-Nagumo (FN) [1] (or real Newell-Whitehead [2], or Kolmogorov-Petrovsky-Piscounov [3]) equation,

$$u_t = u_{xx} + u(1-u)(u-\gamma).$$

This equation admits a "two-shock" solution which describes the interaction of two waves traveling with speed $\pm(2\gamma-1)/\sqrt{2}$ and $\pm(2-\gamma)/\sqrt{2}$ [after interaction, these two waves coalesce and produce a wave traveling with speed $\pm(1+\gamma)/\sqrt{2}$].

Here, we introduce a new method for both deriving nonintegrable equations admitting such solutions, as well as constructing the correpsonding solutions. Our method is based on (a) the introduction of a new type of symmetry, which we call generalized conditional symmetry; (b) a new mathematical characterization of multishock and multisoliton solutions.

We construct nonintegrable equations admitting multishock or multisoliton solutions, by using integrable equations together with certain of their generalized conditional symmetries. Therefore, the origin of the exact solutions admitted by these nonintegrable equations can be traced back to their relationship with integrable equations, and can be thought of as *remnants of integrability*.

We will illustrate our method by discussing classes of nonintegrable equations, which are constructed with the aid of the Burgers, the Korteweg-de Vries (KdV), and the modified KdV equations.

We first introduce the concept of generalized conditional symmetries: The function $\sigma(u)$ is a generalized conditional symmetry (GCS) of the evolution equation $u_t = K(u)$, if

$$K'\sigma - \sigma'K = F(u,\sigma), \quad F(u,0) = 0, \tag{1}$$

where $K(u)$ and $\sigma(u)$ are differentiable functions of $u, u_x, u_{xx}, \ldots$ and $F(u,\sigma)$ is a differentiable function of $u, u_x, u_{xx}, \ldots$ and of $\sigma, \sigma_x, \sigma, \ldots$. The prime denotes the Fréchet derivative, i.e,

$$\sigma'(u) = \frac{\partial\sigma}{\partial u} + \frac{\partial\sigma}{\partial u_x}\partial_x + \frac{\partial\sigma}{\partial u_{xx}}\partial_x^2 + \cdots.$$

If $F(u,\sigma)=0$, then Eq. (1) reduces to the definition of a generalized symmetry [4]. The concept of a conditional symmetry was introduced in [5] under the name of nonclassical symmetry. Generalized conditional symmetries are generalizations of conditional symmetries, in the same way that generalized symmetries are generalizations of symmetries.

The usefulness of this definition follows from the fact that it implies that the equations $u_t = K$ and $\sigma = 0$ are compatible. Therefore, in general, they share a common

0031-9007/94/72(21)/3293(4)$06.00

manifold of solutions. To obtain these solutions one first solves the ordinary differential equation (ODE) $\sigma=0$ to obtain u as a function of x with some x-independent integration constants. One then substitutes this function in the evolution equation $u_t=K$ to determine the time evolution of these constants. This procedure is precisely the same as the one used for obtaining invariant solutions [6]. We will show that a multishock or a multisoliton solution can be characterized by an ODE $\sigma=0$, where σ is a GCS of the integrable equation under consideration.

It also follows from the definition the following important fact: If σ is a GCS of the equation $u_t=K(u)$, and if $G(u,0)=0$, then σ is also a GCS of the equation

$$u_t=K(u)+G(u,\sigma). \tag{2}$$

Furthermore, all these equations share with equation $u_t=K(u)$ the common manifold of solutions obtained through the GCS σ.

Equation (2) implies that the question of constructing nonintegrable equations possessing certain exact solutions reduces to the question of constructing a GCS for some member of the class of equations contained in (2). For integrable equations it is possible to construct GCS in a systematic way. In this Letter we use two methods for constructing GCS. One uses Bäcklund transformation and the other uses a certain reduction of the stationary hierarchy associated with the given integrable equations.

(i) We first discuss the nonintegrable equations constructed from the Burgers hierarchy. This hierarchy is defined by

$$u_t=\sum_{m=0}^{M}\alpha_m\partial_x(\partial_x-u)^m u,\quad M\in Z^+, \tag{3}$$

where $\alpha_0,\alpha_1,\dots,\alpha_M$ are arbitrary constants. Equation (3) with $m=1$, $\alpha_0=0$, $\alpha_1=1$ is Burgers equation $u_t=u_{xx}-2uu_x$. The Burgers hierarchy is usually written as $u_t=\sum_{m=0}^{M}\Phi^m u_x$, where $\Phi=\partial_x-u-u_x\partial_x^{-1}$. Using induction it is easy to show that $\Phi^m\partial_x=\partial_x(\partial_x-u)^m$, which implies Eq. (3). For pedagogical reasons, we first describe how to use Burgers equation to construct equations admitting a two-shock solution. It is well known that the two-shock solution of the Burgers equation satisfies a third order equation,

$$\Phi^2u_x+c_1\Phi u_x+c_2u_x=0. \tag{4}$$

After integrating once, this equation becomes

$$u_{xx}-3uu_x+u^3+c_1(u_x-u^2)+c_2u+c_3=0. \tag{5}$$

The left-hand side (LHS) of Eq. (4) is a symmetry of Burgers equation. It can be verified that the LHS of Eq. (5), which will be denoted by σ, is a GCS of Burgers equation. Therefore, the entire class of Eq. (2), where $K=u_{xx}-2uu_x$ and σ is the LHS of Eq. (5), shares the two-shock solution,

$$u=-\{\ln[K_1e^{\alpha_1x+\alpha_1^2t}+K_2e^{\alpha_2x+\alpha_2^2t}+K_3e^{\alpha_3x+\alpha_3^2t}]\}_x,$$

with the Burgers equation. In this equation, K_1,K_2,K_3, c_1,c_2,c_3 are arbitrary constants, and $\alpha_1,\alpha_2,\alpha_3$ are the roots of $\alpha^3+c_1\alpha^2+c_2\alpha+c_3=0$, which are assumed to be distinct. This class contains the following interesting equations:

$$u_t=uu_x-u^3+\alpha u+\beta, \tag{6}$$

$$u_t=\tfrac{1}{3}u_{xx}+\tfrac{2}{3}(-u^3+\alpha u+\beta), \tag{7}$$

$$u_t=\gamma u_{xx}+(1-3\gamma)uu_x+(1-\gamma)(-u^3+\alpha u+\beta). \tag{8}$$

These equations follow from (2) by taking $G(u,\sigma)=-\sigma,-\frac{2}{3}\sigma,(\gamma-1)\sigma$, respectively. (In these equations $c_1=0$, $c_2=-\alpha$, and $c_3=-\beta$.) Equation (8) was studied by Satsuma [7]. It is interesting to note that first order partial differential equations (PDE's) like Eq. (6) can also support two-shock solutions.

This result can easily be generalized to N-shock solution. Actually the following is valid: The class of equations (2), where K is given by the right-hand side (RHS) of Eq. (3) and σ is defined by

$$\sigma=(\partial_x-u)^Nu+c_1(\partial_x-u)^{N-1}u+\cdots+c_Nu+c_{N+1}$$

($c_1,\dots,c_{N+1}$ are arbitrary constants), admits the N-shock solution

$$u=-\left\{\ln\left[\sum_{n=1}^{N+1}K_n\exp k_n\left(x-\left[\sum_{m=0}^{M}\alpha_mk_n^m\right]t\right)\right]\right\}_x,\quad M\in Z^+.$$

In this equation, $K_1,\dots,K_{N+1}$ are arbitrary constants and $k_1,\dots,k_{N+1}$ are $N+1$ roots of $k^{N+1}+c_1k^N+\cdots+c_Nk+c_{N+1}=0$. The derivation of this result can be found in [8]. A simple consequence of this result is that the lowest order PDE, other than a member of Burgers hierarchy, which admits an N-shock solution is the $(N-1)$st order PDE

$$u_t=\sum_{m=0}^{N-1}\alpha_m\partial_x(\partial_x-u)^mu-\alpha_{N-1}\sigma.$$

For example, the equation

$$u_t=uu_{xx}-3u^2u_x+u^4+\alpha_1(u_{xx}-2uu_x)+\alpha_0u_x-c_1(u_{xx}-3uu_x+u^3)-c_2(u_x-u^2)-c_3u-c_4,$$

admits a three-shock solution.

(ii) Now we discuss the nonintegrable equations constructed from the KdV hierarchy in the potential form. This hierarchy is defined by

$$u_t=\sum_{m=0}^{M}\alpha_m\Phi^mu_x,\quad \Phi=\partial_x^2+8u_x-4\partial_x^{-1}u_{xx},\quad M\in Z^+, \tag{9}$$

where $\alpha_0,\alpha_1,\dots,\alpha_M$ are arbitrary constants. Equation (9) with $m=1$, $\alpha_0=0$, $\alpha_1=1$, is the potential KdV equation $u_t=u_{xxx}+6u_x^2$. The two-soliton solution of this equation satisfies the fifth order ODE,

$$\Phi^2 u_x + c_1 \Phi u_x + c_2 u_x = 0 . \tag{10}$$

The LHS of this equation is a symmetry of the potential KdV equation. Actually, Eq. (10) contains the two-soliton solution as a particular case. The most general solution of Eq. (10) describes the interaction of two-cnoidal waves. It is quite interesting that there exists a second order ODE whose general solution is precisely the two-soliton solution. This ODE is given by

$$u_{xx} - \frac{1}{2u} u_x^2 + 2uu_x + \frac{1}{2} u^3 + \alpha u + \frac{\beta}{u} = 0 . \tag{11}$$

It can be verified that the LHS of this equation, which will be denoted by σ, is a GCS of the potential KdV equation. Therefore, the entire class of equations (2), where $K = u_{xxx} + 6u_x^2$ and σ is the LHS of Eq. (11), shares the two-soliton solution,

$$u = \{\ln[K_1 e^{c_+ x - (2\alpha - 2c_-^2)c_+ t} + K_2 e^{-c_+ x + (2\alpha - 2c_-^2)c_+ t} + K_3 e^{c_- x - (2\alpha - 2c_+^2)c_- t} + K_4 e^{-c_- x + (2\alpha - 2c_+^2)c_- t}]\}_x ,$$

with the potential KdV equation. In this equation, $K_1, K_2, K_3, \alpha, \beta$ are arbitrary constants,

$$c_\pm = (-\alpha \pm \sqrt{\alpha^2 - 2\beta})^{1/2} ,$$

and $K_1 K_2 c_+^2 = K_3 K_4 c_-^2$. This class contains the equations

$$u_t = u_x^2 + 2u^2 u_x - 2\alpha u_x + u^4 + 2\alpha u^2 + 2\beta$$

and

$$u_t = -2uu_{xx} + 2u_x^2 - 2u^2 u_x - 2\alpha u_x .$$

These equations follow from (2) by taking $G(u,\sigma) = -\sigma_x - (u_x/u + u)\sigma$ and $-\sigma_x - (u_x/u + 3u)\sigma$, respectively.

In the case of Burgers equation, it was possible to obtain the GCS (5) by simply integrating Eq. (4). For the case of the KdV, the GCS (11) can be obtained from (10) using the following two methods. First, starting from the ansatz $u_{xx} + g(u, u_x) = 0$ and using repeated differentiation it is possible to show that this equation yields (10) if $g(u, u_x)$ is the expression defined in Eq. (11). Alternatively, Eq. (11) can be derived by using the x part of the Bäcklund transformation of the potential KdV equation [9].

Equation (11) is Painlevé 27 in the classification of Gambier [10]. Under the transformation $u = v_x/v$, it becomes the bilinear ODE

$$v_x v_{xxx} - \tfrac{1}{2} v_{xx}^2 + \alpha v_x^2 + \beta v^2 = 0 .$$

Differentiating this equation, we find the linear ODE

$$v_{xxxx} + 2\alpha v_{xx} + 2\beta v = 0 . \tag{12a}$$

If u satisfies the potential KdV, then v also satisfies a linear evolution equation

$$v_t + 2v_{xxx} + 6\alpha v_x = 0 . \tag{12b}$$

The solution of the two linear equations (12), together with $u = v_x/v$ implies the two-soliton solution.

(iii) We now consider the modified KdV hierarchy

$$u_t = \sum_{m=-M_1}^{M_2} \alpha_m (\partial_x^2 - 4\partial_x u \partial_x^{-1} u)^m u_x , \qquad M_1, M_2 \in Z^+ ,$$

where the α_m's are arbitrary constants. The cases $\alpha_m = \delta_{m,1}$ and $\alpha_m = \delta_{m,-1}$ correspond to the modified KdV equation $u_t = u_{xxx} - 6u^2 u_x$, and to the sinh-Gordon equation $u_t = \frac{1}{2}\sinh(2\partial_x^{-1} u)$, respectively. Using the Miura transformation which relates solutions of this hierarchy to solutions of the KdV hierarchy, it can be shown that the two-soliton solution of the above hierarchy satisfies

$$u_{xx}^2 - 4uu_x u_{xx} + 2u_x^3 - 2u^2 u_x^2 + 6u^4 u_x - 2u^6 - 4k^2 u_x^2 + 8k^2 u^2 u_x - 4k^2 u^4 = 0 . \tag{13}$$

Hence, the entire class of equations (2), where $K = u_{xxx} - 6u^2 u_x$ and σ is the LHS of Eq. (13), share the same two-soliton solution of the modified KdV-sinh-Gordon hierarchy. This class contains the equation

$$u_t = 2uu_{xx} - u_x^2 - 3u^4 + 4k^2 u_x - 4k^2 u^4 .$$

This equation follows from (2) by taking $G(u,\sigma) = \sigma_x/(4uu_x - 2u_{xx})$.

We conclude with several remarks.

(1) An effective way of constructing GCS is to investigate linear first order PDE's. For example, the requirement that

$$\sigma = u_{xx} + (au)_x + f(u), \quad a = a(x,t) ,$$

is a GCS of

$$u_t + (au)_x = 0$$

implies that $f(u)$ is either linear or cubic in u, i.e.,

$$\sigma = u_{xx} + au_x + (a_x + k_1)u + k_2 u^3 ,$$

and a satisfies

$$a_t - 3a_{xx} + 2aa_x = 0 , \quad a_{xt} - a_{xxx} + 2a_x^2 + 3k_1 a_x = 0 ,$$

i.e., $a = -3(\sqrt{k_1/2})\tanh[(\sqrt{k_1/2})x + c]$. The solution corresponding to this GCS can be obtained by noting that the equation $\sigma = 0$ can be transformed to the elliptic equation $d^2w/dz^2 = 2w^3$ and hence can be solved explicitly (for simplicity we assume $k_1 = 1$, $k_2 = -2$),

$$u(x,t) = z_x w(z) , \qquad z(x,t) = c_1 e^{(1/\sqrt{2})x + (3/2)t} + c_2 e^{-(1/\sqrt{2})x + (3/2)t} + c_3 . \tag{14}$$

Using Eq. (2) with $G(u,\sigma)=\sigma$, it follows that this solution is also a solution of the real Newell-Whitehead equation $u_t=u_{xx}+u-2u^3$. Indeed the solution $u(x,t)$ given by (14), where $w(z)$ is the Jacobi elliptic function, was found in [11].

(2) It turns out that the two-shock solution of the Burgers' hierarchy and the one-soliton solution of the potential Caudrey-Dodd-Gibbon-Sawata-Kotera equation,

$$u_t+u_{xxxxx}-30u_xu_{xxx}+60u_x^3=0\,,$$

are characterized by the same GCS. Indeed, using the Bäcklund transformation of the above equation, it can be shown that its one-soliton solution is characterized by $u_{xx}-3uu_x+u^3+\lambda=0$, which is a special case of Eq. (5) [12].

(3) In this Letter we have mostly concentrated on two-soliton solutions. It is of course possible to construct equations possessing higher solitons. However, these equations are more complicated. For example, it can be shown [13] that the ODE

$$[(u_x+u^2)_{xx}+ab+bc+ca]^2-4\{[(u_x+u^2)_x]^2+2abc\}u^2=0\,,$$

where $a=u_x+u^2-k_1^2-k_2^2+k_3^2$, k_1,k_2,k_3 are arbitrary constants, and b,c are obtained from a by cyclic permutation of $k_1\to k_2\to k_3$, characterized the three-soliton solution of the potential KdV. Hence PDE's sharing with potential KdV a three-soliton solution can be constructed.

(4) It is also possible using our method to construct equations sharing with integrable equations other types of solutions. For example, it can be shown [13] that the solution describing the interaction between one soliton and one cnoidal wave is characterized by the ODE

$$\partial_x\{\sqrt{u_{xx}^2-(u_x+k^2+c)[u_{xxx}+2u_x^2+4(k^2+c)(u_x-k^2)]}\}=\tfrac{1}{2}u_{xxx}+3u_x^2+2cu_x\,,$$

where k and c are arbitrary constants. Hence PDE's sharing with potential KdV this type of solution can be constructed.

(5) The concept of GCS provides a new analytical tool of investigating both integrable and nonintegrable equations. The results of some of these investigations will be presented elsewhere. We only note one such result. The Burgers equation admits a conditional symmetry of the type $\sigma=u_x+u^2+\alpha(x,t)u+\beta(x,t)$ if α and β satisfy

$$\alpha_t=\alpha_{xx}-2\alpha\alpha_x+2\beta_x,\quad \beta_t=\beta_{xx}-2\alpha_x\beta\,. \tag{15}$$

These equations are linearizable: Let g be a solution of Burgers equation. For a given g, let φ satisfy the linear equation

$$\varphi_t=\varphi_{xx}-2g\varphi_x+2(g_x+g^2)\varphi\,.$$

Then, the solution of Eqs. (15) is given by $\alpha=-\varphi_x/\varphi$ and $\beta=-g_x-g^2+g\varphi_x/\varphi$.

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