General Relativistic Dynamics of Irrotational Dust: Cosmological Implications

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The nonlinear dynamics of cosmological perturbations of an irrotational collisionless fluid is analyzed within general relativity. Relativistic and Newtonian solutions are compared, stressing the different role of boundary conditions in the two theories. Cosmological implications of relativistic effects, already present at second order in perturbation theory, are studied and the dynamical role of the magnetic part of the Weyl tensor is elucidated.

In a recent paper [1] we have shown that the general relativistic (GR) dynamics of a self-gravitating perfect fluid is greatly simplified under three assumptions: (i) the fluid is collisionless (i.e., with zero pressure, p), (ii) it has zero initial vorticity, ω_{ab} [2], and (iii) the so-called "magnetic" part of the Weyl tensor, H_{ab}, is zero. The former two conditions are wide enough to allow for many cosmological cases, such as the evolution of dark matter adiabatic perturbations generated during inflation. The third assumption is more problematic. In linear theory H_{ab} only contains vector and tensor modes (e.g., Ref. [3]): If the vorticity vanishes no vector modes are present and H_{ab} only contains gravitational waves. Beyond linear theory the meaning of H_{ab} is less straightforward. It is reasonable to assume that $H_{ab}=0$ forbids at least the occurrence of gravitational waves. This is particularly clear in the present context, where, thanks to the absence of pressure gradients, the motion is geodesic and, if H_{ab} also vanishes, no spatial gradients appear in the evolution equations (apart from those contained in convective time derivatives, which can be dropped by going to a comoving frame): It is hard to think of any actual wave propagation with no spatial derivatives appearing in the fluid and gravitational evolution equations.

Following Ellis [4] we describe the dynamics directly in terms of observable fluid and geometric quantities: the mass density ρ, the expansion scalar Θ, and three traceless, flow-orthogonal and symmetric tensors, the shear, σ^a_β, the so-called "electric" part of the Weyl tensor, E^a_β, describing tidal interactions of the fluid element with the surrounding matter, and its magnetic part H^a_β. As noted in Ref. [1], if the magnetic component is switched off, all the equations for the GR dynamics take a strictly local form: Each element evolves independently of the others. Only at the initial time Cauchy data must be consistently given on a spatial hypersurface. The subsequent evolution can be entirely followed in Lagrangian form until caustic formation, when the one-to-one mapping between fluid elements and space points is lost. We call such a system a *silent universe*, in that no information can be exchanged among different fluid elements. This is due to the causal nature of GR, where signal exchange can only occur *dynamically* via gravitational radiation and, in the

case of fluids with nonzero pressure, also via sound waves, but none of these wave modes is allowed when $p=H_{ab}=0$. Because of the advantages of a purely local treatment, this method [1] has recently attracted some attention. In particular, Croudace *et al.* [5] have shown the connection of the GR pancake solution [1] with the Szekeres metric [6]; Bertschinger and Jain [7] have performed a detailed study of the Lagrangian dynamics of fluid elements.

However, the condition $H_{ab}=0$ cannot be taken as an exact constraint for the general cosmological case. It has been shown [8] that the only solutions of Einstein equations, with $p=\omega_{ab}=H_{ab}=0$ are either of Petrov type I, or conformally flat, or homogeneous and anisotropic of Bianchi type I, or *locally* axisymmetric (i.e., with two degenerate shear eigenvalues) and described by a Szekeres line element [6]. All of these cases require some restrictions on the initial data: The exact conditions above are not suitable to study cosmological structure formation. However, requiring $p=\omega_{ab}=0$ and $H_{ab}\approx 0$ appears more feasible. A small H_{ab} is in fact compatible with arbitrary departures from local axisymmetry of fluid elements. This is shown by the behavior of perturbations around Robertson-Walker (RW): Whatever initially scalar perturbations are given, H_{ab} vanishes at first order, but not beyond. A small value of H_{ab} allows arbitrary ratios among the shear eigenvalues, provided the initial perturbations are small. For general initial shapes of the fluid elements the system will radiate gravitationally during nonlinear evolution. However, fully GR numerical computations [9] have shown that only a negligible fraction (less than 1%) of the total energy is carried away in the form of gravitational radiation, during the nonlinear collapse of collisionless ellipsoids. In spite of these facts, as our calculations below demonstrate, a nonzero H_{ab} allows for the influence of the surrounding matter on the evolution of fluid elements. Although this signal travels at finite speed, for perturbations on scales much smaller than the horizon it effectively appears as an instantaneous Newtonian feature. One might wonder whether during the late phases of collapse, when local axisymmetry is expected to be established, the environmental influence on the evolving fluid element can be neglected and the

$H_{ab} = 0$ condition restored.

General relativistic dynamics.—To describe our system we start from the equations of Ref. [4]. We always work in the comoving synchronous gauge $ds^2 = -dt^2 + a^2(t)\tilde{h}_{\alpha\beta}dq^\alpha dq^\beta$, where $a = At^{2/3}$, as for a flat, matter-dominated RW model (our "background" solution). For computational convenience we introduce suitably rescaled quantities: a scaled density fluctuation $\Delta \equiv (6\pi Gt^2\rho - 1)/a$, a peculiar expansion scalar $\vartheta = (3t/2a)(\Theta - 2/t)$, a traceless shear tensor $s_\beta^\alpha \equiv (3t/2a)\sigma_\beta^\alpha$, and a traceless tidal tensor $e_\beta^\alpha \equiv (3t^2/2a)E_\beta^\alpha$. These quantities can be grouped in two spacelike tensors: the velocity gradient tensor $\vartheta_\beta^\alpha \equiv s_\beta^\alpha + \frac{1}{3}\delta_\beta^\alpha\vartheta$, related to the covariant derivatives of the peculiar velocity field; the *peculiar gravitational field* tensor $\Delta_\beta^\alpha \equiv e_\beta^\alpha + \frac{1}{3}\Delta\delta_\beta^\alpha$. We also scale the magnetic tensor as $\mathcal{H}_\beta^\alpha \equiv (3t^2/2a)H_\beta^\alpha$. The dynamical equations for the fluid and the gravitational field are

$$\dot{\vartheta}_\beta^\alpha = -\frac{3}{2a}(\vartheta_\beta^\alpha + \Delta_\beta^\alpha) - \vartheta_\gamma^\alpha\vartheta_\beta^\gamma, \tag{1}$$

$$\dot{\Delta}_\beta^\alpha = -\frac{1}{a}(\vartheta_\beta^\alpha + \Delta_\beta^\alpha) - 2(\vartheta\Delta_\beta^\alpha + \Delta\vartheta_\beta^\alpha) + \frac{5}{2}\Delta_\gamma^\alpha\vartheta_\beta^\gamma + \frac{1}{2}\Delta_\beta^\gamma\vartheta_\gamma^\alpha$$
$$+ \delta_\beta^\alpha(\Delta\vartheta - \Delta_\gamma^\delta\vartheta_\delta^\gamma) + \frac{3t}{4a^2}\tilde{h}_{\beta\eta}(\tilde{\eta}^{\eta\gamma\delta}\mathcal{H}_{\gamma;\delta}^\alpha + \tilde{\eta}^{\alpha\gamma\delta}\mathcal{H}_{\gamma;\delta}^\eta), \tag{2}$$

$$\dot{\mathcal{H}}_\beta^\alpha = -\frac{1}{a}\mathcal{H}_\beta^\alpha - 2\vartheta\mathcal{H}_\beta^\alpha - \delta_\beta^\alpha\vartheta_\delta^\gamma\mathcal{H}_\gamma^\delta + \frac{5}{2}\mathcal{H}_\gamma^\alpha\vartheta_\beta^\gamma + \frac{1}{2}\mathcal{H}_\beta^\gamma\vartheta_\gamma^\alpha$$
$$-\frac{3t}{4a^2}\tilde{h}_{\beta\eta}(\tilde{\eta}^{\eta\gamma\delta}\Delta_{\gamma;\delta}^\alpha + \tilde{\eta}^{\alpha\gamma\delta}\Delta_{\gamma;\delta}^\eta), \tag{3}$$

where the dot denotes partial differentiation with respect to the scale factor a and $\tilde{\eta}^{\alpha\gamma\delta}$ is the Levi-Civita tensor relative to the metric $\tilde{h}_{\alpha\beta}$: $\tilde{\eta}^{\alpha\beta\gamma} = \tilde{h}^{-1/2}\varepsilon^{\alpha\beta\gamma}$, with $\varepsilon^{123} = 1$. The metric tensor evolves according to $\frac{1}{2}\tilde{h}^{\alpha\gamma}\dot{\tilde{h}}_{\gamma\beta} = \vartheta_\beta^\alpha$.

The above tensors have to satisfy the constraints [4]

$$\vartheta_{\alpha;\beta}^\beta = \vartheta_{,\alpha}, \tag{4}$$

$$\Delta_{\alpha;\beta}^\beta = \Delta_{,\alpha} - \frac{2a^2}{3t}\tilde{h}_{\alpha\mu}\tilde{h}_{\beta\nu}\tilde{\eta}^{\mu\lambda\gamma}\vartheta_\lambda^\nu\mathcal{H}_\gamma^\beta, \tag{5}$$

$$\mathcal{H}_{\alpha;\beta}^\beta = \frac{2a^2}{3t}\tilde{h}_{\alpha\mu}\tilde{h}_{\beta\nu}\tilde{\eta}^{\mu\lambda\gamma}\vartheta_\lambda^\nu\Delta_\gamma^\beta, \tag{6}$$

$$\mathcal{H}_\beta^\alpha = \frac{t}{2a}\tilde{h}_{\beta\mu}(\tilde{\eta}^{\mu\gamma\delta}\vartheta_{\gamma;\delta}^\alpha + \tilde{\eta}^{\alpha\gamma\delta}\vartheta_{\gamma;\delta}^\mu). \tag{7}$$

All these are fulfilled at the linear level [3] by growing-mode scalar initial conditions [1]: $\Delta_\beta^\alpha(a_0) = -\vartheta_\beta^\alpha(a_0) = \varphi_{0,\beta}^\alpha$, where the scalar φ_0, an arbitrary function of the space coordinates q^α, is the initial peculiar gravitational potential, related to Bardeen's gauge-invariant Φ_H [10] by $\varphi_0 = -(3/2A^3)\Phi_H$. These initial conditions correspond to the "seed" metric $\tilde{h}_{\alpha\beta} = \delta_{\alpha\beta}(1 - \frac{20}{9}A^3\varphi_0) - 2a\varphi_{0,\alpha\beta}$, and imply vanishing initial $\mathcal{H}_\beta^\alpha$ (the constant mode, $\propto A^3\varphi_0 \ll 1$, can be neglected in practice, compared to the growing mode $\propto a\varphi_{0,\alpha\beta}$).

The Lagrangian dynamics is determined by Eqs. (1), (2), and (3) plus the initial data. One obtains a local Eu-

lerian description of the fluid [1], using the "generalized Hubble law" [4]. We have $\xi^\alpha = \vartheta_\beta^\alpha\xi^\beta$, where $a\xi^\alpha$ is the infinitesimal spatial displacement of neighboring elements. The matrix connecting the Eulerian coordinates x^α with the Lagrangian ones q^β is the Jacobian $J_\beta^\alpha \equiv \partial x^\alpha/\partial q^\beta \equiv \delta_\beta^\alpha + \mathcal{D}_\beta^\alpha$, where $\mathcal{D}_\beta^\alpha$ is the (symmetric) deformation tensor. Taking $\xi^\alpha = dx^\alpha = J_\beta^\alpha\xi_{(0)}^\beta$, where $\xi_{(0)}^\beta = dq^\beta$ represent the initial (i.e., Lagrangian) infinitesimal displacements, one gets $\dot{\mathcal{D}}_\beta^\alpha = \vartheta_\beta^\alpha + \vartheta_\gamma^\alpha\mathcal{D}_\beta^\gamma$, formally solved by $\mathcal{D}_\beta^\alpha(a) = \exp\int_{a_0}^a d\bar{a}\vartheta_\beta^\alpha(\bar{a}) - \delta_\beta^\alpha$. Once the Jacobian is known one gets the metric as $\tilde{h}_{\alpha\beta} = \tilde{h}_{\gamma\delta}(a_0)J_\alpha^\gamma J_\beta^\delta$. As shown in Refs. [1,8], if $\mathcal{H}_\beta^\alpha = 0$, the tensors $\vartheta_\beta^\alpha, \Delta_\beta^\alpha, \tilde{h}_{\alpha\beta}$ commute and they can be diagonalized simultaneously. In such a case, Eqs. (1) and (2) can be reduced to six first-order equations for the six eigenvalues of ϑ_β^α and Δ_β^α. Along the local principal axes we can set $\tilde{h}_{\alpha\beta} = \delta_{\alpha\beta}\tilde{h}_\beta$ and $\vartheta_\beta^\alpha = \delta_\beta^\alpha\vartheta_\beta$ and get $\tilde{h}_\alpha(a) = \tilde{h}_\alpha(a_0)\exp2\int_{a_0}^a d\bar{a}\vartheta_\alpha(\bar{a})$. In the locally axisymmetric case, i.e., when two eigenvalues of $\varphi_{0,\beta}^\alpha$ coincide, a relation exists with particular Szekeres solutions [6].

Newtonian dynamics.—The equations which govern the nonlinear dynamics of a collisionless fluid in Newtonian theory (NT) for an expanding universe [11] can be written in suitably rescaled form as (e.g., Ref. [12])

$$\dot{u}^\alpha + u^\beta u_{,\beta}^\alpha = -\frac{3}{2a}(u^\alpha + \varphi_{,}^\alpha), \tag{8}$$

$$\dot{\Delta} + u^\beta\Delta_{,\beta} = -\frac{1}{a}(u_{,\beta}^\beta + \Delta) - \Delta u_{,\beta}^\beta, \tag{9}$$

$$\varphi_{,\beta}^\beta = \Delta, \tag{10}$$

where φ is the peculiar gravitational potential. Differentiating the Euler equation (8), defining the symmetric tensors $\vartheta_\beta^\alpha \equiv u_{,\beta}^\alpha$, with $u^\alpha = dx^\alpha/da$, and $\Delta_\beta^\alpha \equiv \varphi_{,\beta}^\alpha$, and adopting a Lagrangian description, one recovers Eq. (1), while the continuity equation (9) coincides with the trace of Eq. (2). It is clear that the NT is degenerate, as it provides only one equation to determine the tensor Δ_β^α: any traceless tensor added to the right-hand side of Eq. (2), leaves the NT equations unchanged. In order to completely determine the evolution of the gravitational field tensor Δ_β^α one has to resort to its definition in terms of the potential φ, i.e., to a *nonlocal* theory. Because of the intrinsic nonlocality of NT [the Poisson equation (10) is an elliptic, constraint equation] one needs boundary conditions to determine the dynamics: Contrary to the GR equations, initial data are not enough. It is well known (e.g., Ref. [4]) that the lack of evolution equations for the traceless part of the gravitational tensor, e_β^α, implies that the NT adds spurious solutions which would be discarded by the full GR system.

Beyond the Zel'dovich approximation.—In order to see the behavior of the GR solutions and evaluate the role of the magnetic term we construct a second order Lagrangian perturbation expansion in the amplitude of the fluctuations around RW. It will prove useful to define the two quantities $\mu_1 \equiv \varphi_{0,\gamma}^\gamma = \lambda_1 + \lambda_2 + \lambda_3$ and $\mu_2 = \frac{1}{2}(\varphi_{0,\gamma}^\gamma\varphi_{0,\delta}^\delta$

$-\varphi\delta_{,\delta}\varphi\delta_{0,\gamma}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, where λ_a are the local eigenvalues of the symmetric tensor $\varphi^{\alpha}_{0,\beta}$. One immediately obtains the traces, $\vartheta = -\mu_1 + a(-\mu_1^2 + \frac{8}{7}\mu_2)$ and $\Delta = \mu_1 + a(\mu_1^2 - \frac{4}{7}\mu_2)$, which coincide with those obtained in Lagrangian second-order NT [13]. After very lengthy calculations we obtain

$$\vartheta^{\alpha}_{\beta} \equiv -\varphi^{\alpha}_{0,\beta} + \frac{a}{7}(-12\mu_1\varphi^{\alpha}_{0,\beta} + 6\mu_2\delta^{\alpha}_{\beta} + 5\varphi^{\alpha}_{0,\gamma}\varphi^{\gamma}_{0,\beta}) + \chi^{\alpha}_{\beta}$$

(11)

(here indices are raised by the Kronecker symbol), having kept only growing modes. The expressions for Δ^{α}_{β} and $\mathcal{H}^{\alpha}_{\beta}$ will not be reported here for shortness. The traceless tensor χ^{α}_{β}, representing the contribution due to the magnetic part, has zero divergence: $\chi^{\alpha}_{\beta,\alpha} = 0$. It can be written as a convolution $\chi^{\alpha}_{\beta}(\mathbf{q},a) = \int d^3q' S^{\alpha}_{\beta}(\mathbf{q}')f(|\mathbf{q} - \mathbf{q}'|,a)$, of the source $S^{\alpha}_{\beta} = \mu^{\alpha}_{2,\beta} - \nabla^2(2\mu_1\varphi^{\alpha}_{0,\beta} - 2\varphi^{\alpha}_{0,\gamma}\varphi^{\gamma}_{0,\beta} - \delta^{\alpha}_{\beta}\mu_2)$ with the function f, whose Fourier transform $\hat{f}(k)$ satisfies the equation

$$\hat{f}''' + \frac{9}{\tau}\hat{f}'' + \frac{12}{\tau^2}\hat{f}' + k^2\left[\hat{f}' + \frac{3}{\tau}\hat{f}\right] = \frac{10A^3\tau}{21} ,$$

(12)

where a prime denotes differentiation with respect to the conformal time $\tau = (3/A)t^{1/3}$. The initial conditions are $\hat{f}(\tau_0) = \hat{f}'(\tau_0) = \hat{f}''(\tau_0) = 0$. Asymptotic solutions of Eq. (12), confirmed by a numerical check, are $\hat{f} \approx 2A^3\tau^2/21k^2$ for $k\tau \gg 1$ and $\hat{f} \approx A^3\tau^4/378$ for $k\tau \ll 1$. Performing a second-order expansion for the deformation tensor and defining $\mathcal{D}^{\alpha}_{\beta} \equiv -a\varphi^{\alpha}_{0,\beta} + a^2\psi^{\alpha}_{\beta}$, we find

$$\psi^{\alpha}_{\beta} = \frac{1}{7}(-2\mu_1\varphi^{\alpha}_{0,\beta} + \mu_2\delta^{\alpha}_{\beta} + 2\varphi^{\alpha}_{0,\gamma}\varphi^{\gamma}_{0,\beta}) + \frac{1}{a^2}\int_{a_0}^{a}d\bar{a}\chi^{\alpha}_{\beta} ,$$

(13)

with trace $\psi^{\alpha}_{\alpha} = -\frac{3}{7}\mu_2$. The symmetric tensor ψ^{α}_{β} provides the second-order correction to the deformation tensor, whose first order is the kinematical Zel'dovich approximation. The metric tensor reads

$$\tilde{h}_{\alpha\beta} = \delta_{\alpha\beta} - 2a\varphi_{0,\alpha\beta} + \frac{a^2}{7}(19\varphi_{0,\alpha\gamma}\varphi^{\gamma}_{0,\beta} - 12\mu_1\varphi_{0,\alpha\beta} + 6\mu_2\delta_{\alpha\beta})$$

$$+ \int_{a_0}^{a}d\bar{a}\chi_{\alpha\beta} .$$

(14)

We then have $dx^{\alpha} = dq^{\alpha} - a\varphi^{\alpha}_{0,\beta}dq^{\beta} + a^2\psi^{\alpha}_{\beta}dq^{\beta}$. In NT one would write the same formal expression, but the irrotationality condition would lead to $\psi^{\alpha}_{\beta} = \psi^{\alpha}_{,\beta}$, with the potential ψ satisfying the second-order Poisson equation [13] $\nabla^2\psi = -\frac{3}{7}\mu_2$, which is consistent with the trace of the GR equation. In other words, the NT eigenvalues v_a of ψ^{α}_{β} only need to satisfy the condition $\Sigma_a v_a = -\frac{3}{7}\mu_2$. In order to get the complete information on the single v_a's one needs the NT definition of ψ^{α}_{β} as $\psi^{\alpha}_{,\beta}$, i.e., a nonlocal information. The GR v_a's also solve the NT equations, but the reverse is not necessarily true: it depends upon the boundary conditions used in solving Poisson's equation.

Inside the horizon.—Suppose that the source, hence $\varphi^{\alpha}_{0,\beta}$, has some typical scale of variation l, i.e., $l \sim \varphi^{\alpha}_{0,\beta}/\varphi^{\alpha}_{0,\beta\gamma}$. If $l \ll \tau$ we find $\vartheta^{\alpha}_{\beta} \approx -\varphi^{\alpha}_{0,\beta} - a\varphi^{\alpha}_{0,\gamma}\varphi^{\gamma}_{0,\beta} + 2a\psi^{\alpha}_{,\beta}$. The second-order deformation tensor reduces to $\psi^{\alpha}_{\beta} = \psi^{\alpha}_{,\beta}$, while the metric reads $\tilde{h}_{\alpha\beta} = \delta_{\alpha\beta} - 2a\varphi_{0,\alpha\beta} + a^2\psi_{,\alpha\beta}$. All these expressions coincide with those of second-order NT and can be obtained from the $c \to \infty$ limit of Eq. (12). The scalar ψ carries information on the influence of the surrounding matter on the dynamics of fluid elements. Note that $\psi^{\alpha}_{,\beta}$ produces a tilt of the principal axes of the first-order deformation tensor, $\varphi^{\alpha}_{0,\beta}$.

Outside the horizon.—When $l \gg \tau$, $\chi^{\alpha}_{\beta} \approx (3t^2/14a)S^{\alpha}_{\beta}$, and the contribution to $\vartheta^{\alpha}_{\beta}$ due to the magnetic term becomes negligible. The relevant expressions can be obtained from Eqs. (11), (13), and (14) with $\chi^{\alpha}_{\beta} \approx 0$. Perturbations with size greater than the Hubble radius evolve as a separate *silent* universe: spatial gradients play no role in this case. However, these local GR effects have little cosmological implications, since perturbations on super-horizon scales usually have very small amplitude, and a linear approximation is sufficient. Nevertheless, there are a number of formal consequences, which are worth mentioning. One of these is the *absence of 2D solutions.* If one eigenvalue of $\varphi^{\alpha}_{0,\beta}$, e.g., λ_3, vanishes everywhere, the NT, with suitable boundary conditions, implies $\vartheta_3(a) = 0$ or $x_3(a) = q_3$, i.e., no motion along the third axis. This is referred to as "two-dimensional" (2D) gravitational clustering. As far as the second-order deformation tensor is concerned, one would have $v_1 + v_2 = -\frac{3}{7}\mu_2$, with $\mu_2 = \lambda_1\lambda_2$, and $v_3 = 0$. In the GR case, instead, we find $v_1 = v_2 = -v_3 = -\frac{3}{7}\mu_2$, and $\vartheta_3(a) \neq 0$ for $a \neq a_0$. The motion dynamically impressed along the third axis soon becomes of the same order of magnitude as that in the other directions. This effect is due to the tide-shear coupling terms $\delta^{\alpha}_{\beta}(\vartheta\Delta - \Delta^{\delta}_{\gamma}\vartheta^{\gamma}_{\delta})$ in the evolution equation for Δ^{α}_{β}, which reduces to $-2\mu_2\delta^{\alpha}_{\beta}$ to lowest order. The only case when this coupling disappears is when two λ_a's simultaneously vanish, i.e., for planar symmetry. Therefore $\vartheta_3(a) = 0$ is not an exact solution of the GR equations, unless another ϑ_a also vanishes. As an example, no axisymmetric configurations without motion along the symmetry axis are allowed.

This discussion leads to the main issue: The general nonlinear dynamics of fluid elements. So far, two analytical solutions of our system are known: For planar configurations, $\lambda_1 = \lambda_2 = 0$, one recovers the Zel'dovich pancake solution, as shown in Ref. [1]; for exactly spherical configurations, $\lambda_1 = \lambda_2 = \lambda_3$, the local solution is the well-known top-hat model (e.g., Ref. [11]). Croudace *et al.* [5] looked for solutions representing attractors among the trajectories of our system with zero magnetic tensor. They found that both spherical collapse and a perfect pancake are repellers for general initial conditions, and argued that the pancake instability is probably due to having disregarded the contribution of $\mathcal{H}^{\alpha}_{\beta}$. On the other hand, Bertschinger and Jain [7] have shown that the in-

stability of the pancake solution is caused by the tide-shear coupling in the evolution of the tide, which tends to destabilize the pancake solution (for general initial conditions) but would stabilize prolate configurations. For vanishing $\mathcal{H}^\alpha_\beta$, a strongly prolate spindle with expansion along its axis is the generical outcome of collapse, except for specific initial conditions corresponding to exactly spherical or planar configurations. Our analysis shows that the dynamical effect causing preferential collapse to expanding spindles in the $\mathcal{H}^\alpha_\beta=0$ case is the GR tide-shear coupling in the tide evolution equation. This term is not present in NT, although it is compatible with its equations. This is further illustrated by the collapse of an infinite homogeneous ellipsoid ($l\to\infty$), which is described by our equations with zero $\mathcal{H}^\alpha_\beta$. As is well known, the NT dynamics favors the formation of oblate spheroids (e.g., Ref. [14]), pancakelike objects with one collapsing axis and the other two tending to a finite size (apart from initial conditions corresponding to an initial prolate spheroid). The GR collapse (e.g., Ref. [15]) favors the formation of prolate spheroids, collapsing filaments with expansion along their symmetry axis. However, as our second-order calculations show, the evolution of fluid elements as isolated ellipsoids does not apply to perturbations on scales smaller than the Hubble radius: here nonlocal effect play a fundamental role. The actual nonlinear dynamics would generally result from the competition of the *local* GR tide-shear coupling, causing pancake instability, and the nonlocal environmental influence, carried by the magnetic part of the Weyl tensor. During the early deviations from linear evolution, as described by Lagrangian second-order perturbation theory, the latter effect dominates; however, extending this conclusion to the late strongly nonlinear phases would require further study.

Finally, as a result of this analysis, we are able to calculate how many gravitational waves are produced within a second-order approximation (remember that at this order the magnetic tensor is traceless and transverse, so it is related to gravitational radiation [3]). Outside the horizon $H_{\alpha\beta}\approx(4a^3/7t)(\varepsilon_{\alpha\gamma\delta}\varphi^\delta_{0,\beta}\varphi^\gamma_{0,\nu}+\varepsilon_{\beta\gamma\delta}\varphi^\delta_{0,\alpha}\varphi^\gamma_{0,\nu})^{,\nu}$, while in-

side the horizon $H_{\alpha\beta}$ decays as $1/a$. Only a tiny amount of gravitational waves is produced on subhorizon scales at this level; however, their dynamical role is far from being negligible.

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