Multifractal Structure of Multiplicity Distributions in Particle Collisions at High Energies

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Experimental data on the bin size dependence of charged hadron multiplicity distributions in proton-antiproton collisions and electron-positron annihilation are analyzed in terms of multifractals. Linear relations are found between $\langle n \ln n\rangle/\langle n\rangle$ and $\ln\langle n\rangle$, and also between $\ln\langle n^q\rangle$ and $\ln\langle n\rangle$ for $q=2,3,4,\ldots$, where n is the multiplicity in a single bin of the (pseudo) rapidity space and $\langle\ \rangle$ stands for the event average. Generalized dimensions D_q for $q=0,1,2,\ldots$ are determined from the slopes. Our result suggests a power law of the form $\langle n^q\rangle = c(\langle n\rangle/n_0)^{b_q}$.

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The very high multiplicity of hadrons produced in particle and nuclear collisions at very high energies makes it feasible to study fluctuations of nonstatistical origin [1,2]. In their pioneering works, Miyamura and co-workers analyzed the multiplicity distributions in limited intervals of rapidity using the normalized factorial moments and emphasized the importance of bin size dependence [3]. Bialas and Peschanski analyzed the bin size dependence of the normalized factorial moment of the JACEE events and found evidence for nonstatistical fluctuations called intermittency [4]. Intermittency has then been studied extensively and stimulated other related approaches. In particular, approaches based on the concept of (multi)fractals seem to be most interesting as they may be related to phase transitions [5], self-similar cascades, chaos, entropy, etc. In fact, some interesting results have already been obtained by again analyzing the JACEE events [6]. However, it has often happened that experimental data do not show the expected linear behavior in a log-log plot. This may be at least partially due to the fact that most methods are unable to give the required mathematical limit; the number of points $\to\infty$. The purpose of this paper is to propose a new method [7] that overcomes this difficulty and to present a successful result of an application to UA5 data on proton-antiproton collisions [8] and TASSO and DELPHI data on electron-positron annihilation [9,10].

General formalism.— Consider a process of multiparticle production at some incident energy and the distribution in the rapidity (y) space. A single event contains n hadrons distributed in the interval $y_{\min} < y < y_{\max}$. The multiplicity n changes from event to event according to the distribution $P_n(y)$ where $y = y_{\max} - y_{\min}$. Divide the full rapidity interval of length y into ν bins of equal size $\delta y = y/\nu$. The multiplicity distribution for a single bin is denoted as $P_n(\delta y)$ for $n=0,1,2,3,\ldots$, where we assume that the inclusive rapidity distribution dn/dy is constant and $P_n(\delta y)$ is independent of the location of the bin. Hadrons produced in Ω independent events are distributed in $\Omega\nu$ bins of size δy. Let K be the total number of hadrons produced in these Ω events and n_{ai} the multiplicity of hadrons in the ith bin of the ath event. The theory of multifractals [11] motivates us to consider the normalized density p_{ai} defined by

$$p_{ai} = n_{ai}/K \tag{1}$$

and to consider if the quantity

$$T_q(\delta y) = \ln \sum_{a=1}^{\Omega} \sum_{i=1}^{\nu} p_{ai}^q \quad \text{for } q>0 \tag{2}$$

behaves like a linear function of the logarithm of the "resolution" $R(\delta y)$,

$$T_q(\delta y) = A_q + B_q \ln R(\delta y)\,, \tag{3}$$

where A_q and B_q are constants independent of δy. If such a behavior is observed for a considerable range of $R(\delta y)$, a generalized dimension may be determined as

$$D_q = \frac{B_q}{q-1}\,. \tag{4}$$

Here the case with $q=1$ is obtained by taking an appropriate limit [11]. The result is equivalent to consider entropy defined by

$$S(\delta y) = -\sum_{a=1}^{\Omega} \sum_{i=1}^{\nu} p_{ai} \ln p_{ai} \tag{5}$$

and to see if it behaves as

$$S(\delta y) = -\sigma \ln R(\delta y) + \text{const}\,, \tag{6}$$

where $\sigma = D_1$ is the information dimension.

Now we evaluate the double sum of p_{ai}^q. For sufficiently large Ω, one has

$$\sum_{a=1}^{\Omega} \sum_{i=1}^{\nu} p_{ai}^q = \sum_{n=0}^{\infty} \Omega\nu P_n(\delta y) \left(\frac{n}{K}\right)^q = \frac{\langle n^q\rangle}{K^{q-1}\langle n\rangle}\,, \tag{7}$$

where a generic notation

$$\langle f(n)\rangle = \sum_{n=0}^{\infty} f(n) P_n(\delta y) \tag{8}$$

and an obvious relation $\Omega\nu = K/\langle n\rangle$ has been used, $\langle f(n)\rangle$ being a function of δy in general but the δy dependence is suppressed for brevity. Since $\langle n\rangle = K\delta y/\Omega Y$, comparison of (7) with (3) and (4) indicates a relation

$$\begin{aligned}\ln\langle n^q\rangle &= A_q + (B_q+1)\ln\delta y \\ &= A_q + \{(q-1)D_q + 1\}\ln\delta y\end{aligned} \tag{9}$$

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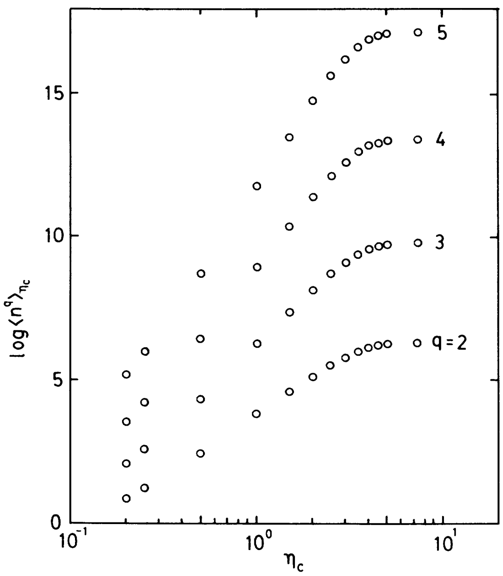

FIG. 1. $\ln\langle n^q\rangle$ vs η_c for $\bar{p}p$ collisions at $\sqrt{s}=200$ GeV.

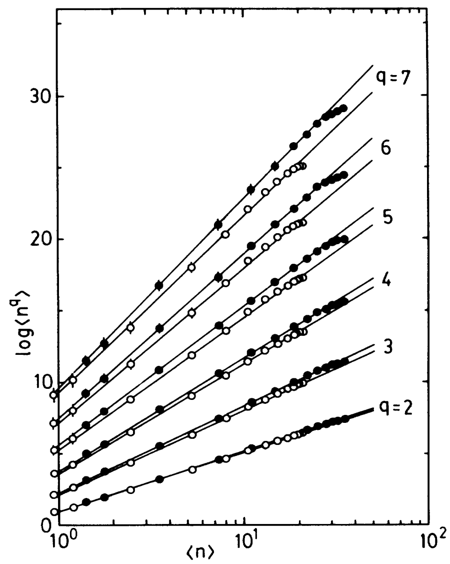

FIG. 2. $\ln\langle n^q\rangle$ vs $\langle n\rangle$ for $\bar{p}p$ collisions at $\sqrt{s}=200$ GeV (open circles) and 900 GeV (solid circles) with linear fits. For each q, a few data points at the largest $\langle n\rangle$ region are neglected in the fit as they deviate systematically from the linear behavior.

for the simplest choice $R(\delta y)=\delta y$.

The information entropy given by (5) can be worked out in a similar manner:

$$S(\delta y)=-\sum_{n=0}^{\infty}\Omega\nu P_n(\delta y)\frac{n}{K}\ln\frac{n}{K}$$
$$=-\frac{\langle n\ln n\rangle}{\langle n\rangle}+\ln K\,. \tag{10}$$

Note that the entropy defined above is entirely different from another entropy $\tilde{S}(\delta y)=-\sum_n P_n(\delta y)\ln P_n(\delta y)$ which was studied by Simak, Sumbera, and Zborovsky [12]. Comparing (10) with (6), one expects a linear relation

$$\frac{\langle n\ln n\rangle}{\langle n\rangle}=D_1\ln\delta y+C \tag{11}$$

for the same choice of $R(\delta y)$.

Data analysis.—Available experimental data on charged hadron multiplicity distributions $P_n(\delta y)$ are mostly taken for a symmetric bin $-y_c<y<y_c$ (or $-\eta_c<\eta<\eta_c$) of the center of mass (pseudo)rapidity discarding the information from the outer region. Our general formalism can be applied to such data with no difficulty. The task is to calculate $\langle n\ln n\rangle/\langle n\rangle$ and $\ln\langle n^q\rangle$ for such a symmetric interval and to see if they behave like a linear function of $\ln y_c$ or not.

UA5 data on $\bar{p}p$ collisions are given in two forms: C_q moment defined by $C_q=\langle n^q\rangle/\langle n\rangle^q$ and parameter fit using the negative binomial distribution (NBD). As the C_q moment is useless to calculate $\langle n\ln n\rangle$, we have used the NBD fit to calculate both $\langle n\ln n\rangle$ and $\langle n^q\rangle$. (We have confirmed that $\langle n^q\rangle$ calculated from the two kinds of data agree within the experimental errors.) The result for $\ln\langle n^q\rangle$ as a function of η_c is shown in Fig. 1. Although the data points with intermediate η_c lie approximately on a straight line, considerable deviation is seen in the larger η_c region. A similar deviation is seen also in a plot of $\langle n\ln n\rangle/\langle n\rangle$ vs $\ln\eta_c$ (not shown). It is reasonable to expect that this deviation may be related to the nonflat behavior of $dn/d\eta$ in the large η region (the projectile or target fragmentation region). Then we expect that $\langle n\rangle=\langle n(\eta_c)\rangle$ may be a better choice of $R(\eta_c)$. Note that $dn/d\langle n\rangle$ is flat by definition. [We would like to note that the utility of the variable $\langle n(\eta_c)\rangle$ in this context was suggested by us in the analysis of the η_c dependence of $\tilde{S}(\eta_c)$ [13] and independently by Bialas and Gazdzicki in their intermittency analysis [14].] The corresponding plot is shown in Fig. 2. A considerable improvement is achieved and now there is no difficulty in determining the slope of the linear fit,

$$\ln\langle n^q\rangle=A_q+\{(q-1)D_q+1\}\ln\langle n\rangle\,. \tag{12}$$

A similar improvement is obtained also for $\langle n\ln n\rangle/\langle n\rangle$ if plotted against $\ln\langle n\rangle$ as shown in Fig. 3, where the data points obtained from e^+e^- data by TASSO [9] are also plotted. The data points from $\bar{p}p$ at $\sqrt{s}=200$ GeV and 900 GeV lie on different straight lines while those from e^+e^- annihilation at different energies (14–43.6 GeV) lie on a single straight line indicating a kind of approximate scaling. Those lines are represented as

$$\langle n\ln n\rangle/\langle n\rangle=C_1+D_1\ln\langle n\rangle\,. \tag{13}$$

TASSO and DELPHI data [9,10] also show an approximate linear behavior in a $\ln\langle n^q\rangle$ vs $\ln\langle n\rangle$ plot. Thus we can determine D_q for both $\bar{p}p$ and e^+e^-. The results are shown in Fig. 4, where a trivial but remarkable re-

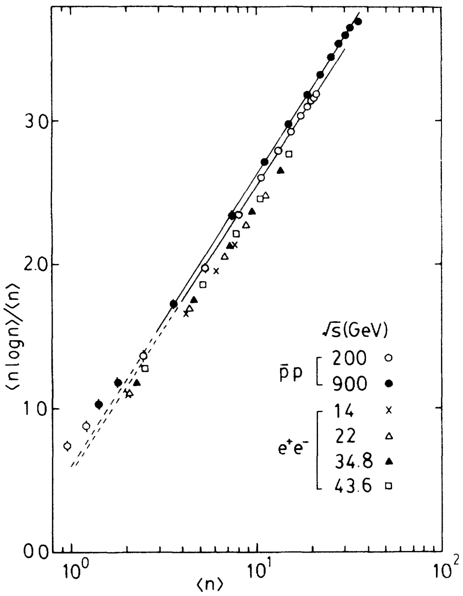

FIG. 3. $\langle n \ln n\rangle/\langle n\rangle$ vs $\langle n\rangle$ for both $\bar{p}p$ collisions and e^+e^- annihilation with linear fits. A few data points in the small $\langle n\rangle$ region are omitted in the fit.

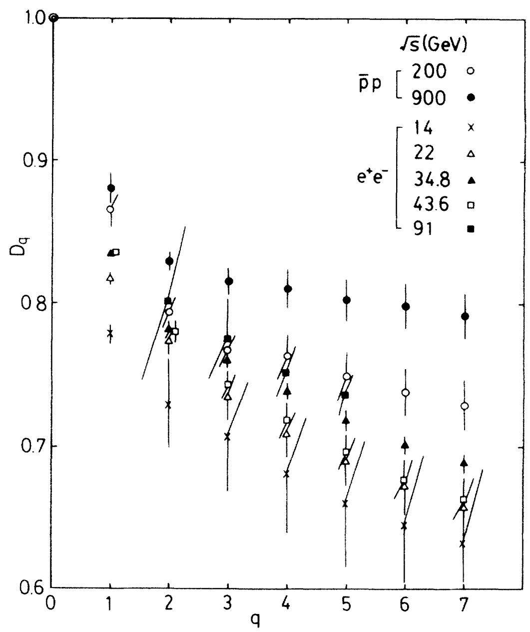

FIG. 4. Generalized dimension as a function of q for both $\bar{p}p$ collisions and e^+e^- annihilation at various energies. The point $D_0=1$ indicated by a double circle refers to both processes at every energy.

sult $D_0=1$ is also shown. If $\langle n\rangle$ is sufficiently large, $P_0(\delta y)\ll 1$ and hence $\langle n^0\rangle \sim 1$. Equation (12) then gives $D_0=1$. It should be noted here that an erroneous result $D_0=0$ is obtained if (12) is used in the small $\langle n\rangle$ limit.

It is formally possible to extend the analysis to negative q if empty bins are omitted in the summation. However, we found that the $\ln\langle n^q\rangle$ vs $\ln\langle n\rangle$ plots for $\bar{p}p$ collisions and e^+e^- annihilation for $q=-1$–-6 do not show a linear behavior.

If one extrapolates the linear curves fitted to $\ln\langle n^q\rangle$ data down to the smaller $\langle n\rangle$ region, they cross each other in a rather narrow region of the $\ln\langle n^q\rangle$-$\ln\langle n\rangle$ plane. Therefore, it is quite probable that they actually cross each other at a single point:

$$(\ln\langle n^q\rangle,\ln\langle n\rangle)=(\Gamma,\Delta)$$
$$=(-2.16\pm 0.08,-3.40\pm 0.30)\,. \qquad (14)$$

Equation (14) with (12) implies a power law,

$$\langle n^q\rangle = c\left(\frac{\langle n\rangle}{n_0}\right)^{(q-1)D_q+1} \quad \text{for } q=2,3,4,\ldots, \qquad (15)$$

where $c=e^\Delta$ and $n_0=e^\Gamma$. An even simpler case with $c=n_0$ may be realized eventually because in that case, (15) includes a trivial identity for $q=1$.

It is a well-known experimental fact that the full phase space multiplicity distribution in e^+e^- annihilation is much narrower than the one in $\bar{p}p$ or generic hadron-hadron collisions. The broader distribution in hadron-hadron collisions is usually attributed to a dynamical, structural, or geometrical effect, e.g., fluctuations in the number of strings, parton momenta, or the impact parameter. Therefore, our result suggests that the multifractal structure revealed by our method has a universal origin which is not very sensitive to those effects.

It is easy to verify that $\langle n \ln n\rangle/\langle n\rangle$ cannot be negative. Therefore, the linear relation (13) should cease to hold for too small $\langle n\rangle$. As is seen in Fig. 3, the first few data points at small $\langle n\rangle$ already show the expected deviation.

The result of our multifractal analysis is summarized as follows. (i) Both $\langle n \ln n\rangle/\langle n\rangle$ and $\ln\langle n^q\rangle$ ($q=2,3,4,\ldots$) in $\bar{p}p$ collisions and e^+e^- annihilation exhibit a remarkable linear behavior as functions of $\ln\langle n\rangle$. (ii) The generalized dimension D_q determined from the slope of the linear-log behavior has the following systematics: (a) $1>D_q(\bar{p}p)>D_q(e^+e^-)$ for $q>0$; (b) starting from a trivial value $D_0=1$, D_q decreases monotonically as q increases in accord with the theory of multifractals [11]; (c) D_q for a given reaction tends to increase slowly as $\sqrt{s}$ increases. (iii) Both $\bar{p}p$ and e^+e^- data suggest a simple power law $\langle n^q\rangle=c(\langle n\rangle/n_0)^{b_q}$, $b_q=(q-1)D_q+1$ with the possibility $c=n_0$. (iv) As both $\bar{p}p$ collisions and e^+e^- annihilation exhibit a similar multifractal structure, it may not be attributed to fluctuations of geometrical origin [15].

Some final remarks. Our approach is different from both the intermittency approach [4,16] and the later version of the G-moment approach [17]. A crucial difference lies in the fact that the total number of points (particles) K in our approach can be made arbitrarily large by taking the total number of events Ω arbitrarily large. See Eq. (1). In the other two approaches, a finite multiplicity per event (or its average) appears in the corresponding denominator, making it impossible to take the required mathematical limit. On the other hand, there is a resemblance between our approach and the early version of the G-moment approach [18]. However, important differences lie in the facts that we include empty bins for positive q, for varying δy we do not fix $\langle n\rangle\Omega$ (which corresponds to K in Ref. [18] where it is fixed as $\delta y \to 0$), and we do not require δy to be small (we, rather, require $\langle n\rangle$ to be large enough) in the linear-log fit and have a distinct result $D_0 = 1$.

The multifractal structure revealed by the present analysis represents a remarkable property of the observed fluctuations. No attempt has been made to separate the fluctuations into statistical and nonstatistical ones. However, we have confirmed that a simple statistical model with Poissonian $P_n(\delta y)$ gives a trivial result $D_q = 1$ for $q = 1,2,3,\ldots$ in the large $\langle n\rangle$ limit. The difference $1 - D_q$ is a measure of nonstatistical fluctuations. Anyway, this problem deserves further study and more details will be given elsewhere.

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