Lightlike Wilson Loops and Gauge Invariance of Yang-Mills Theory in 1+1 Dimensions

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A lightlike Wilson loop is computed in perturbation theory up to $O(g^4)$ for pure Yang-Mills theory in 1+1 dimensions, using Feynman and light-cone gauges to check its gauge invariance. After dimensional regularization in intermediate steps, a finite gauge invariant result is obtained, which, however, does not exhibit Abelian exponentiation. Our result is at variance with the common belief that pure Yang-Mills theory is free in 1+1 dimensions, apart perhaps from topological effects.

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While an abundant literature is by now available concerning "Euclidean" QCD in two dimensions [1–3] comparatively fewer investigations have been performed in Minkowskian 1+1 dimensions. It is a common belief, originating from the pioneering work of 't Hooft [4], that Yang-Mills theory (YMT) without fermions in two dimensions is a free theory apart perhaps from topological effects. As a matter of fact, the gauge field should not be endowed with any dynamical degree of freedom. This can naively be seen in axial gauges, where one of the components of the vector potential is set equal to zero. It is also at the root of the possibility of calculating the mesonic spectrum in the large N approximation [4,5], when quarks are introduced.

However, the theory exhibits severe infrared (IR) divergences which need to be regularized. In Ref. [4] an explicit IR cutoff is advocated, which turns out to be uninfluent in the bound state equation; a Cauchy principal value (CPV) prescription in handling such IR singularity leads indeed to the same result [6]. Still, difficulties in performing a Wick's rotation in those conditions have been pointed out [7], and a causal prescription for the IR singularity has been advocated, leading to a quite different solution for the vector propagator. In this context the bound state equation with vanishing bare quark masses [8] has solutions with quite different properties, when compared with the ones of Refs. [4,5].

In view of the above-mentioned controversial results and of the fact that "pure" YMT does not immediately look free in Feynman gauge, where degrees of freedom of a "ghost" type are present, we have thought it worthwhile to perform a test on a gauge invariant quantity. We have chosen a rectangular Wilson loop with lightlike sides, directed along the vectors $n_\mu = (T, -T)$ and $n^*_\mu = (L, L)$, parametrized according to the equations

$$\begin{aligned} C_1 &: x^\mu(t) = n^{*\mu} t, \\ C_2 &: x^\mu(t) = n^{*\mu} + n^\mu t, \\ C_3 &: x^\mu(t) = n^\mu + n^{*\mu}(1-t), \\ C_4 &: x^\mu(t) = n^\mu(1-t), \qquad 0 \le t \le 1. \end{aligned} \tag{1}$$

This contour has been considered in Refs. [9,10] for an analogous test of gauge invariance in 1+3 dimensions. Its lightlike character forces a Minkowski treatment.

We shall perform a perturbative calculation up to $O(g^4)$; topological effects will not be considered. We can anticipate the unexpected results we will obtain: the gauge invariant theory is not free at d=1+1, at variance with the commonly accepted behavior; the theory in $d = 1 + (D-1)$ is "discontinuous" in the limit $D \to 2$.

The calculation in Feynman gauge.—In Feynman gauge already the free vector propagator does not exist as a tempered distribution in 1+1 dimensions. A regularization is thereby mandatory and we choose from here on to adopt the dimensional regularization, which preserves gauge invariance:

$$D^F_{\mu\nu}(x) = -g_{\mu\nu}\frac{\pi^{-D/2}}{4}\Gamma(D/2-1)(-x^2+i\epsilon)^{1-D/2}. \tag{2}$$

The calculation in Feynman gauge of the lightlike Wilson loop (1), in $1 + (D-1)$ dimensions up to $O(g^4)$ has been performed in Ref. [9] and will not be repeated here. Actually in Ref. [9] part of the contribution from graphs containing three vector lines has only been given as a Laurent expansion around $D = 4$, owing to its complexity. In the following we shall exhibit its general expression in terms of a generalized hypergeometric series and then we shall expand it around $D = 2$. We only report

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for the reader's benefit the final results concerning the contributions of the various diagrams.

Single vector exchange $[O(g^2)]$ gives

$$W_F^{(1)} = -\left(\frac{\hat{g}}{\pi\mu}\right)^2 C_F \frac{\Gamma(D/2-1)}{(D-4)^2}\mathcal{C}, \quad (3)$$

where $\hat{g}^2 = g^2(\mu^2)^{D/2-1}$, μ being the running renormalization mass, C_F the Casimir operator of the fundamental representation of color su(N), and

$$\mathcal{C} \equiv [(2\pi\mu^2 nn^* + i\epsilon)^{2-D/2} + (-2\pi\mu^2 nn^* + i\epsilon)^{2-D/2}]. \quad (4)$$

One immediately notices that the propagator pole at $D = 2$ is canceled after integration over the contour, leading to a finite result.

At $O(g^4)$ we can restrict ourselves to the so-called "maximally non-Abelian" contributions [11]. The self-energy correction to the propagator gives

$$W_F^{(2;se)} = \left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{64} \frac{\Gamma^2(D/2-1)(3D-2)}{(D-4)^3(D-3)(D-1)}\mathcal{A}, \quad (5)$$

where C_A is the Casimir operator of the adjoint representation,

$$\mathcal{A} \equiv [(2\pi\mu^2 nn^* + i\epsilon)^{4-D} + (-2\pi\mu^2 nn^* + i\epsilon)^{4-D}] \quad (6)$$

and the fermionic loop has not been considered (pure YMT). Equation (5) exhibits a double pole at $D = 2$.

Next we consider the contribution of the so-called "cross" graphs, the ones with two noninteracting crossed vector exchanges,

$$W_F^{(2;\mathrm{cr})} = -\left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{16} \frac{\Gamma^2(D/2-1)}{(D-4)^4} \times \left[\mathcal{A} + 8\mathcal{B}\left(1 - \frac{\Gamma^2(3-D/2)}{\Gamma(5-D)}\right)\right], \quad (7)$$

where

$$\mathcal{B} \equiv [(2\pi\mu^2 nn^* + i\epsilon)(-2\pi\mu^2 nn^* + i\epsilon)]^{2-D/2}. \quad (8)$$

Again a double pole occurs at $D = 2$.

The contribution coming from graphs with three vector lines is by far the most complex one. It is convenient to split it into two parts, one coming from graphs with two vector lines attached to the same side,

$$W_F^{(2;\mathrm{ss})} = -\left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{16} \frac{\mathcal{A}}{(D-4)^4} \left[\frac{\Gamma^2(D/2-1)}{D-3} - 2\Gamma(3-D/2)\Gamma(D/2-1)\Gamma(D-3)\right], \quad (9)$$

and another one in which the three "gluons" end in three different rectangle sides

$$W_F^{(2;\mathrm{ds})} = \left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{64} \mathcal{A} \left\{\frac{\Gamma^2(D/2-2)\Gamma(4-D/2)}{\Gamma(D/2)} \times \Gamma(D-3)F(D) - \frac{4}{(D-4)^4}\left[\Gamma^2(D/2-1) - \Gamma(3-D/2)\Gamma(D/2-1)\Gamma(D-3)\right]\right\}. \quad (10)$$

The function $F(D)$ is defined as

$$F(D) = S(D) + \frac{D/2-1}{(3-D/2)(D-4)}[5\psi(3-D/2) - \psi(D/2-1) - 2\psi(1) - 2\psi(5-D)], \quad (11)$$

$\psi(D)$ being the digamma function and $S(D)$ the convergent generalized hypergeometric series,

$$S(D) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \frac{1}{n!} \frac{\Gamma(n+D-3)}{\Gamma(D-3)} \frac{\Gamma(n+4-D/2)}{\Gamma(4-D/2)} \times \frac{\Gamma(D/2)}{\Gamma(n+D/2)}. \quad (12)$$

Both contributions exhibit a double pole at $D = 2$. The Laurent expansion of Eq. (10) around $D = 4$ reproduces exactly the expression given in Ref. [9].

Summing Eqs. (5), (7), (9), and (10) and performing a careful Laurent expansion around $D = 2$, it is tedious but straightforward to prove that double and single poles cancel, leaving only the finite contribution

$$W_F^{(2)}(D=2) = \left(\frac{g^2}{4\pi}\right)^2 C_F C_A (n^* n)^2 \left(1 + \frac{\pi^2}{3}\right). \quad (13)$$

The presence of a nonvanishing $C_F C_A$ contribution is a dramatic result: it means that the theory does not exponentiate in an Abelian way, as a "bona fide" free theory should do. In order to better understand this result, it is worth turning our attention now to the same Wilson loop calculation, performed in the light-cone axial gauge $nA = 0$.

The calculation in light-cone gauge $nA = 0$.—The free vector propagator in light-cone gauge is very sensitive to the prescription used to handle the so-called "spurious" singularity. The only prescription known so far which allows one to perform a Wick's rotation without extra terms and to calculate loop diagrams in a consistent way [12] is the causal Mandelstam-Leibbrandt (ML) prescription [13]. In a canonical formalism it is obtained by imposing equal time commutation relations [14]; in two dimensions a "ghost" degree of freedom still survives, as will be discussed in the last section.

When the ML prescription is adopted, the free vector propagator is indeed a tempered distribution at $D = 2$ [15], at variance with its behavior in Feynman gauge. In particular, when $x_\perp = 0$,

$$n^{*\mu} n^{*\nu} D_{\mu\nu}^{LC}(x) = \frac{2\pi^{-D/2}\Gamma(D/2)}{4-D} \frac{(xn^*)^2}{(-x^2+i\epsilon)^{D/2}}. \quad (14)$$

The calculation of the Wilson loop under consideration at $O(g^4)$ in $1+(D-1)$ dimensions, using light-cone gauge, has been performed in Ref. [10]. Here we shall report those results and then perform their Laurent expansion around $D=2$, the value we are interested in.

One might wonder why dimensional regularization should be introduced at all, as one might presume that single graph contributions are likely to be finite in this gauge. On the other hand, while remaining strictly at $D=2$, no self-interaction should be present.

We shall discuss this point of view at the end of the paper. For the time being let us recall that, when $D \neq 2$, "transverse" vector components are turned on and, although their contribution is expected to be $O(D-2)$, it can compete with singularities arising from loop corrections. This is indeed what happens in the self-energy calculation, as will be soon apparent.

The calculation $O(g^2)$ is easily performed and the result exactly coincides with Eq. (3), for any value of D. At $O(g^4)$ we again confine ourselves to the "maximally non-Abelian" contributions, without losing information. The self-energy graph now gives

$$W_{LC}^{(2;\mathrm{se})} = \left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{16} \mathcal{A} \left\{ \frac{4}{(4-D)^4(D-3)} \times \left[\frac{\Gamma^2(3-D/2)\Gamma(D-3)}{\Gamma(5-D)} - \Gamma^2(D/2-1) \right] + \frac{\Gamma^2(D/2-1)}{(4-D)^3(D-3)} \left[3 - \frac{3D-2}{4(D-1)} - \frac{D-2}{D-3} \right] \right\}. \tag{15}$$

Its limit at $D=2$,

$$W_{LC}^{(2;\mathrm{se})}(D=2) = \left(\frac{g^2}{4\pi}\right)^2 C_F C_A (n^* n)^2, \tag{16}$$

is finite, but it does not vanish, as one might have naively expected. Similarly the contribution from the "cross" graph

$$W_{LC}^{(2;\mathrm{cr})} = -\left(\frac{\hat{g}}{\pi\mu}\right)^4 \frac{C_F C_A}{16} \frac{\Gamma^2(D/2-1)}{(D-4)^4} \left\{ 2\mathcal{A}\frac{D-2}{D-3} + 8\mathcal{B}\left[1 - 2\frac{\Gamma^2(3-D/2)}{\Gamma(5-D)}\right] \right\} \tag{17}$$

leads to a finite, nonvanishing, result in the limit $D=2$,

$$W_{LC}^{(2;\mathrm{cr})}(D=2) = \left(\frac{g^2}{4\pi}\right)^2 C_F C_A (n^* n)^2 \frac{\pi^2}{3}. \tag{18}$$

Summing Eqs. (16) and (18) we exactly recover Eq. (13). As a matter of fact the contribution due to graphs with three "gluon" lines [10],

$$W_{LC}^{(2;3g)} = \Omega \left\{ \Gamma(D/2-2)\Gamma(3-D/2) + \frac{\Gamma^2(3-D/2)}{\Gamma(5-D)} \times \frac{6D-28}{(D-2)(D-4)} - \frac{2}{\Gamma(2-D/2)} S_1(D) - (4-D)\Gamma(3-D/2) S_2(D) \right\}, \tag{19}$$

where

$$\Omega = \frac{2g^4 C_F C_A}{(2\pi)^D} (2nn^*)^{4-D} e^{\frac{-i\pi D}{2}} \cos\left(\frac{\pi D}{2}\right) \frac{\Gamma(D-4)}{(D-4)^2}, \tag{20}$$

$$S_1(D) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2-D/2)}{(n+3-D/2)n!} \left[\psi(n+D/2) - \psi(n+3-D/2) + \frac{2}{(n+D/2-1)(n+D/2)} - \frac{\Gamma(n+D/2-1)\Gamma(5-D)}{\Gamma(4+n-D/2)} + \frac{1}{n+3-D/2} \right], \tag{21}$$

and

$$S_2(D) = \sum_{n=0}^{\infty} \frac{\Gamma(n+3-D/2)}{\Gamma(n+6-D)} \left[\Gamma(D/2-2) \times \left(\frac{\Gamma(n+5-D)}{\Gamma(n+3-D/2)} - \frac{\Gamma(n+2)}{\Gamma(n+D/2)} \right) + 2\frac{\Gamma(n+1)\Gamma(D/2)}{\Gamma(n+1+D/2)} + \frac{\Gamma(n+5-D)}{\Gamma(n+4-D/2)} \times \Gamma(D/2-1) - \frac{\Gamma(n+1)\Gamma(3-D/2)}{\Gamma(n+4-D/2)} \right], \tag{22}$$

vanishes when $D=2$.

As a consequence the same finite result for the Wilson loop $O(g^4)$ at $D=2$ is obtained both in Feynman and in light-cone gauges. However, non-Abelian terms are definitely present; the theory cannot be considered a free one in quantum loop calculations at $D=2$, in spite of the quadratic nature of its classical Lagrangian density in light-cone gauge. From a practical viewpoint, in this fully interacting theory, the hope of getting solutions, when quarks are included, e.g., for the mesonic spectrum, in analogy with 't Hooft's treatment, seems to us remote.

The 't Hooft approach.—In this section we stick to 1+1 dimensions. If we interpret x^- as time direction, the field A_- is not an independent dynamical variable and just provides a nonlocal force of Coulomb type between fermions. In momentum space it can be described by the "exchange" k_+^{-2} [4]. Owing to its singular IR behavior, this expression is not a tempered distribution; however, it can be Fourier transformed after an analytical regularization,

$$D_{--}(x) = \frac{i}{(2\pi)^2} \int e^{ikx} d^2k |k_+|^{-2\lambda} \Big|_{\lambda=1}. \tag{23}$$

Alternatively the same result can be obtained by interpreting the square as (minus) the derivative of the Cauchy principal value (CPV) distribution

$$D_{--}(x) = -\frac{i}{(2\pi)^2}\int e^{ikx}d^2k\frac{\partial}{\partial k_+}\left[\mathrm{CPV}\left(\frac{1}{k_+}\right)\right] = -\frac{i}{2}|x_-|\delta(x_+). \quad (24)$$

It is straightforward to check that, by inserting Eq. (24) in our Wilson loop, the result (3) at $O(g^2)$ is recovered.

At $O(g^4)$ in 1+1 dimensions, the only *a priori* surviving non-Abelian contribution, which is due to "cross" graphs, vanishes using Eq. (24). Henceforth no $C_F C_A$ term appears, in agreement with Abelian exponentiation, but at variance with the result obtained (after regularization) in Feynman gauge. On the other hand, no fully consistent vector loop calculation would be feasible in $1+(D-1)$ dimensions, using a CPV prescription or introducing IR cutoffs [16].

If we perform instead an equal time canonical quantization, starting from the Lagrangian density

$$L = \tfrac{1}{2}F^a_{+-}F^a_{+-} + \lambda^a n A^a, \quad (25)$$

λ^a being Lagrange multipliers, by imposing the equal time commutation relations,

$$[A^a_1(t,x), F^b_{01}(t,y)] = i\delta(x-y)\delta^{ab}, \quad (26)$$

we recover for the vector propagator exactly the ML prescription restricted at $D=2$.

In this context the equation for the Lagrange multipliers

$$n\partial\lambda^a = 0 \quad (27)$$

is to be interpreted as a true equation of motion and the fields λ^a provide propagating degrees of freedom, although of a "ghost" type [14]. The potentials A^a_- have the momentum decomposition

$$\tilde{A}^a_-(k) = u^a\delta'(k_+) + v^a\delta(k_+), \quad (28)$$

$\tilde{\lambda}^a\ (k)$ being proportional to u^a: $\tilde{\lambda}^a = k_- u^a$.

The canonical algebra (26) induces on u^a and v^a the algebra

$$[v^a_\pm(k_-), u^b_\mp(q_-)] = \pm\delta(k_- - q_-)\delta^{ab}, \quad (29)$$

$v^a_\pm$ and $u^a_\pm$ being defined as

$$v^a(k_-) = \theta(k_-)v^a_+(k_-) + \theta(-k_-)v^a_a(-k_-),$$
$$u^a(k_-) = \theta(k_-)u^a_+(k_-) - \theta(-k_-)u^a_-(-k_-),$$

all others commutators vanishing. This algebra eventually produces the propagator (14) for $D=2$. No wonder then that we recover from the "cross" graphs Eq. (18), whereas, in strictly 1+1 dimensions, neither self-energy corrections nor graphs with three vector lines should be considered.

The result we obtain in this third scenario neither coincides with the one in Feynman gauge (the limit $D \to 2$ being "discontinuous"), as there is no self-energy contribution in strictly 1+1 dimensions, nor obeys Abelian exponentiation as in 't Hooft's approach, the reason being rooted in a different content of the degrees of freedom (the fields λ^a). Although perhaps more satisfactory from a mathematical viewpoint [7], it looks in our opinion less coherent and its "physical" interpretation looks somewhat obscure.

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