Heat-Flow Induced Anomalies in Superfluid ^{4}He near T_λ

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We study the effect of a stationary heat current Q on superfluid ^{4}He in a homogeneous metastable state near $T_\lambda(Q)$. On the basis of a renormalization-group calculation we predict a sizable enhancement of the specific heat $C_P(Q)$ and a weak depression of the superfluid density $\rho_s(Q)$ up to a critical heat current $Q_c(T)$ where C_P and ρ_s exhibit cusplike anomalies.

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Instabilities of macroscopic systems under stationary nonequilibrium conditions constitute a fundamental area of research in condensed matter physics. Unlike critical phenomena in thermal equilibrium such instabilities often exhibit mean-field type characteristics since fluctuation effects tend to be suppressed by the externally driven flow through the system. Here we shall address ourselves to an interesting instability where fluctuations remain important although the system is not in thermal equilibrium: the superfluid transition of ^{4}He in the presence of a finite heat current Q. This transition occurs at a depressed transition temperature

$$T_\lambda(Q) = T_\lambda(0)[1-(Q/Q_0)^x], \qquad (1)$$

where x is an exponent of a non-mean-field type. The measured value [1] is $x = 0.813 \pm 0.012$; it differs somewhat from the prediction [2,3] $x = [(d-1)\nu]^{-1} = 0.744$ in $d = 3$ dimensions, with $\nu = 0.672$ being the correlation-length exponent [4].

While ^{4}He above $T_\lambda(Q)$ is in a nonequilibrium state with an inhomogeneous temperature profile [3,5], superfluid ^{4}He at finite Q below $T_\lambda(Q)$ can still exist in a state with a homogeneous temperature T [6], provided that Q is sufficiently small. This is a metastable state since it can decay via the nucleation of vortex rings [7]. Here we shall assume the smallness of Q only in the sense that this nucleation rate and the ensuing temperature gradient are negligible. Then, as T approaches $T_\lambda(Q)$ from the homogeneous superfluid side, the transition at $T = T_\lambda(Q)$ may be interpreted [8] as an analog of the instability at the spinodal line of a first-order transition.

It is not obvious *a priori* to what extent fluctuation effects at this nonequilibrium transition are important. The aim of this Letter is to perform a renormalization-group (RG) analysis of this problem focused upon well observable quantities: the specific heat and the superfluid density. On the basis of a RG calculation without adjustment of parameters we predict a novel fluctuation effect of sizable magnitude on the specific heat $C_P(T,Q)$ for $T \lesssim T_\lambda(Q)$ whereas the superfluid density $\rho_s(T,Q)$ is found to deviate only weakly from its equilibrium value $\rho_s(T,0)$. Our results imply that both quantities exhibit cusplike anomalies at $T = T_\lambda(Q)$.

It is well known [6] that a heat current Q in superfluid ^{4}He creates a superfluid counterflow with normal and superfluid velocities v_n and v_s according to $Q = \rho_s S T(v_n - v_s)$ where S is the entropy per unit mass. Near T_λ this relation may be approximated by $Q = -S_\lambda T_\lambda \rho_s v_s$ or

$$Q = -g_0 k_B T_\lambda J_s, \qquad (2)$$

where $J_s = \mathrm{Im}\langle\psi^* \nabla\psi\rangle$ is the superfluid current, k_B is Boltzmann's constant, and $g_0 = 2.16 \times 10^{11}$ sec^{-1}. Following [7] we describe the superfluid state at finite superfluid velocity $\mathbf{v}_s = \hbar\mathbf{k}/m_4$ by the probability distribution $p\{\psi(\mathbf{x})\} \sim \exp[-H\{\psi(\mathbf{x})\}]$ with

$$H\{\psi(\mathbf{x})\} = \int d^dx \left[\frac{r_0}{2}|\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + u_0|\psi|^4\right], \qquad (3)$$

where $\psi(\mathbf{x})$ is the complex order-parameter field. For finite $\mathbf{k}$ and $r_0 < 0$ the mean-field equation $\delta H/\delta\psi = 0$ has the solution [9] $\psi_{\mathrm{mf}}(\mathbf{x}) = \eta_{\mathrm{mf}} \exp i\mathbf{k}\cdot\mathbf{x}$ with

$$\eta_{\mathrm{mf}}^2 = -(r_0 + k^2)/4u_0. \qquad (4)$$

Our basic approximation is to neglect the creation of vortex configurations by performing a perturbation expansion around ψ_{mf} at a given $\mathbf{k}$. This will lead to an order parameter of plane-wave structure

$$\langle\psi(\mathbf{x})\rangle = \eta(r_0,k)\exp i\mathbf{k}\cdot\mathbf{x}, \qquad (5)$$

with $\eta(r_0,k) \neq \eta_{\mathrm{mf}}$ corresponding to a state with uniform $\mathbf{v}_s = \hbar\mathbf{k}/m_4$. The homogeneous temperature T enters via

$$r_0 = r_{0c} + a_0 t, \quad t = (T - T_\lambda)/T_\lambda, \qquad (6)$$

where T_λ is the transition temperature at $Q = 0$. We are primarily interested in the constant-pressure specific heat per unit volume C_P whose definition [10–12] can be generalized to the case of finite $\mathbf{k}$ according to

$$C_P = C_B - \frac{1}{2}a_0^2 \frac{\partial}{\partial r_0}\langle|\psi|^2\rangle(r_0,k), \qquad (7)$$

$$\langle|\psi|^2\rangle(r_0,k) = 2\frac{\partial}{\partial r_0}\Gamma(\eta;r_0,k)\big|_{\eta=\eta(r_0,k)}, \qquad (8)$$

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with the background contribution C_B. Here $\Gamma(\eta,r_0,k)$ is the generating functional of vertex functions (per unit volume) [13] evaluated for the plane-wave order parameter Eq. (5). The amplitude $\eta(r_0,k)$ is determined by $\partial\Gamma/\partial\eta=0$. The perturbation expansion yields

$$\langle|\psi|^2\rangle(r_0,k)=\eta_{\rm mf}^2-2I(r_0,k)\,, \quad (9)$$

$$I(r_0,k)=(2\pi)^{-d}\int d^dp[p^2+c_0^2-4k^2\cos^2\Theta]^{-1}\,, \quad (10)$$

where Θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, and $c_0^2=-2(r_0+k^2)$. The one-loop integral $I(r_0,k)$ can be evaluated analytically (using dimensional regularization at infinite cutoff [13]) by means of the integral representation of the hypergeometric function $F(a,b;c;z)$ [14]. This leads to

$$I(r_0,k)=-\frac{1}{\varepsilon}A_d c_0^{d-2}F\left(\frac{2-d}{2},\frac{1}{2};\frac{d}{2};4k^2/c_0^2\right)\,, \quad (11)$$

with $\varepsilon=4-d$ and $A_d=\Gamma(3-d/2)[2^{d-2}\pi^{d/2}(d-2)]^{-1}$. The resulting bare specific heat, Eq. (7), needs to be renormalized in order to appropriately account for the effect of the critical fluctuations near the superfluid transition at $k=0$ and $r_0=r_{0c}$. We employ the minimal subtraction scheme at fixed $d<4$ [10,12,15]. The Z factors remain unchanged since no new ultraviolet divergences arise at $\mathbf{k}\neq 0$; thus we have the renormalized parameters $u=\mu^{-\varepsilon}Z_u^{-1}Z_\psi^2A_du_0$ and $r=Z_r^{-1}(r_0-r_{0c})$. This implies that the well-known structure of the renormalized specific heat below T_λ [10,12] $C=Z_m^{-1}C_B^{-1}C_P$ is maintained,

$$C=1+\gamma^2F_-(u,r/\mu^2,k/\mu)\,, \quad (12)$$

and that the finite-$\mathbf{k}$ effect is completely contained in the amplitude function F_-. The parameter γ is determined by [12] $\gamma^2=\mu^{-\varepsilon}Z_m^{-1}Z_r^{-2}A_dC_B^{-1}a_0^2/4$. Our result for F_- reads

$$F_-(u,r/\mu^2,k/\mu)=(2u)^{-1}-8/\varepsilon+4(d-2)(c/\mu)^{-\varepsilon}\Phi(z)/\varepsilon\,, \quad (13)$$

$$\Phi(z)=F\left(\frac{2-d}{2},\frac{1}{2};\frac{d}{2};z\right)+(z/d)F\left(\frac{\varepsilon}{2},\frac{3}{2};\frac{d+2}{2};z\right)\,, \quad (14)$$

with $c^2=-2(r+k^2)$ and $z=4k^2/c^2$. We see that a finite superfluid velocity $v_s=\hbar k/m_4$ causes a nontrivial fluctuation effect on the specific heat which appears in leading order in the one-loop term of Eq. (13) whereas the mean-field contribution $(2u)^{-1}$ remains independent of k. Application of the RG theory [10,12,13] implies effective parameters $u(l)$, $\gamma(l)$, and $r(l)$ whose flow parameter $l(t)$ can be related to $t<0$ via $-2r(l)=\mu^2l^2$ both at $k=0$ and $k\neq 0$; hence F_- enters the result for C_P in the form $F_-[u(l),-\frac{1}{2},k/\mu l]$.

We turn directly to the application of our results to $d=3$ dimensions in the asymptotic region $-t<10^{-3}$ where $u(l)\approx u(0)\equiv u^*=0.0362$ and $\gamma(l)^2\sim l^{-\alpha/\nu}$ with $\alpha=2-3\nu<0$ [10,15,16]. Then the physical (bare) specific heat becomes

$$C_P(T,\kappa)=B+\tilde{A}[(4\nu/\alpha)+F_-(\kappa)](-t)^{-\alpha}\,, \quad (15)$$

$$F_-(\kappa)=(2u^*)^{-1}-8+2\kappa^{-1}\arcsin[2\kappa(1-2\kappa^2)^{-1/2}]\,, \quad (16)$$

with $F_-(\kappa)\equiv F_-(u^*,-\frac{1}{2},\kappa)$ and $\kappa\equiv k\xi(-2t)$ where $\xi(t)=\xi_0t^{-\nu}$ is the correlation length above T_λ with $\xi_0=1.4$ Å. At $k=0$ the one-loop result [10] $F_-(0)=(2u^*)^{-1}-4$ is recovered. The constants B and $\tilde{A}$ can be related to the specific heat at $k=0$ [4],

$$C_P(T,0)=B+(A^-/\alpha)(-t)^{-\alpha}\,. \quad (17)$$

The as yet missing link between κ, T, and Q is established via Eq. (2) where now J_s is considered as a function of T and κ according to

$$J_s(T,\kappa)=\xi(-2t)^{1-d}f_J(\kappa)\,. \quad (18)$$

The scaling function f_J is known in one-loop order and is shown in Fig. 4 of Ref. [8]. Inverting Eqs. (2) and (18) yields $\kappa=\kappa(T,Q)$. The final result for the specific heat as a function of T and Q,

$$C_P[T,Q]=C_P(T,\kappa(T,Q))\,, \quad (19)$$

is obtained after substitution of $\kappa(T,Q)$ into Eq. (15).

Similarly we can obtain the superfluid density as a function of T and Q,

$$\rho_s[T,Q]=\rho_s(T,\kappa(T,Q))\,, \quad (20)$$

from the known [8] result (at $d=3$),

$$\rho_s(T,\kappa)=\text{const}\,\xi(-2t)^{-1}\kappa^{-1}f_J(\kappa)\,. \quad (21)$$

Both functions $F_-(\kappa)$ and $f_J(\kappa)$ become complex for $\kappa>\kappa_c=1/\sqrt{6}$ and exhibit the cusplike behavior $\sim-(\kappa_c-\kappa)^{1/2}$ and $\sim(\kappa_c-\kappa)^{3/2}$, respectively, for $\kappa\lesssim\kappa_c$. These singularities do not exist at the mean-field level.

The instability at $\kappa=\kappa_c$ corresponds to the critical superfluid velocity $v_{sc}(T)=(\hbar/m_4)\kappa_c\xi(-2t)^{-1}$ and implies a critical heat current $Q_c(T)$ or a transition temperature $T_\lambda(Q)$, Eq. (1), determined by $\kappa(T,Q_c)=\kappa_c$ or $\kappa(T_\lambda(Q),Q)=\kappa_c$, respectively. Only for $T<T_\lambda(Q)$ or $Q<Q_c(T)$ can the system exist in the metastable homogeneous state for which our theory predicts novel heat-flow induced fluctuation effects as shown in Fig. 1 for several Q. We see that the heat current causes a sizable enhancement of C_P and a weak depression of ρ_s up to $T_\lambda(Q)$ where C_P and ρ_s exhibit cusplike singularities. These effects can be represented in the scaling forms,

$$C_P[T,Q]-C_P[T,0]=(-t)^{-\alpha}f(Q/Q_c)\,, \quad (22)$$

$$\rho_s[T,Q]=\rho_s[T,0]f_\rho(Q/Q_c)\,, \quad (23)$$

for $0\le Q\le Q_c(T)$. The scaling functions are shown in Fig. 2. We note that at the mean-field level $f_\rho(Q/Q_c)$ would remain nonsingular for $Q/Q_c=1$ and f would vanish for all Q. On the basis of quantitative experience with

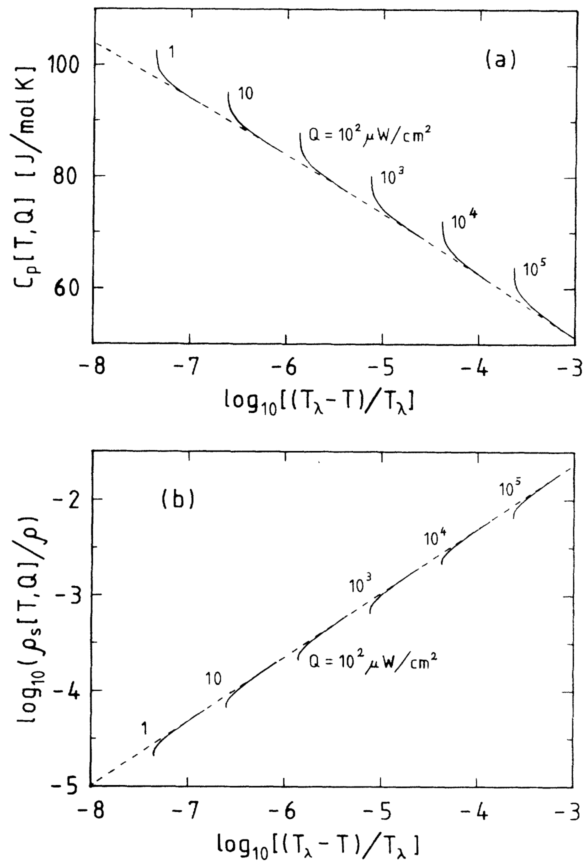

FIG. 1. RG predictions (solid lines) of the specific heat $C_P[T,Q]$ (a), Eqs. (15) and (19), and of the superfluid fraction $\rho_s[T,Q]/\rho$ (b), Eqs. (20) and (21), vs reduced temperature $-t$ for various Q. The dashed lines corresponding to $Q=0$ represent (a) Eq. (17) and (b) $\rho_s/\rho = k_0(-t)^\zeta$ taken from Ref. [4]. The solid lines terminate at $T_\lambda(Q)$ with finite values and infinite slopes.

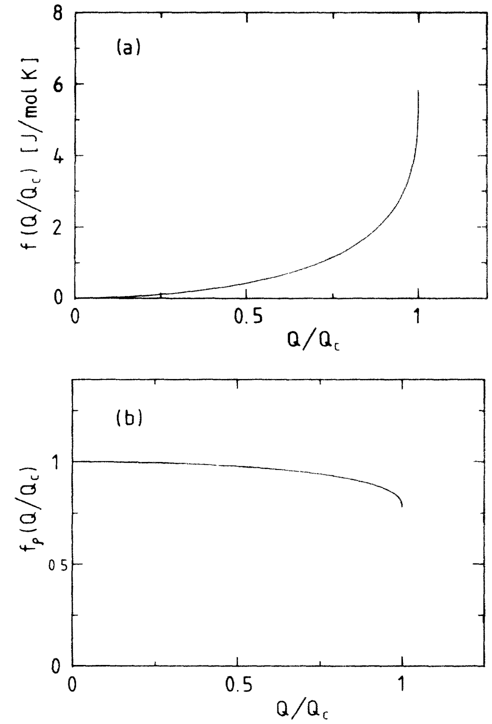

FIG. 2. RG predictions of the scaling functions of the specific heat (a), Eq. (22), and of the superfluid density (b), Eq. (23), vs $Q/Q_c(T)$.

field theory at $k=0$ [16,17] we expect that our one-loop approximation for the scaling functions f and f_ρ is reasonably good. We cannot exclude, however, the possibility that the type of singularity at Q_c is modified at higher order.

It would be interesting to test the predicted scaling functions experimentally. Corresponding measurements are planned for future research [18]. It remains to be seen whether $T = T_\lambda(Q)$ can be approached sufficiently closely, i.e., whether the experiments at finite Q can be performed such that the influence of vortex nucleation is negligible [19]. This may be realizable in some temperature range below $T_\lambda(Q)$ where the predicted enhancement of the specific heat is already observable. Experiments under microgravity conditions may be advantageous in that they could explore the small-Q regime where vortex generation is expected to be less important.

It should be noted that, in addition to the static quantities C_P and ρ_s, there are also transport properties that exhibit interesting finite-Q effects near $T_\lambda(Q)$: the nonlinear boundary (Kapitza) resistance $R_K(T,Q)$ [20–22] and the anisotropic propagation of second sound [23,24]. As far as the latter is concerned, an instability is predicted [24] to occur at $T_\lambda(Q)$ or $Q_c(T)$ where the velocity c_2 and the damping D_2 vanish for waves propagating parallel to **Q**. As far as R_K is concerned, no theoretical explanation for the observed crossover temperature $T_c(Q)$ [20] between a "linear" and a "nonlinear" region is as yet available. A different nonlinear-Q effect, however, has been predicted [22] in the form of a divergence of $R_K(T,Q)$ at some $T^*(Q)$ comparable to but slightly smaller than $T_\lambda(Q)$. It remains to be seen whether in future experiments on the Q dependence of C_P and ρ_s (in a cell of finite thickness) the possible disturbance caused by the boundary resistance can be made negligibly small.

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