

## Leading Quantum Correction to the Newtonian Potential

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I argue that the leading quantum corrections, in powers of the energy or inverse powers of the distance, may be computed in quantum gravity through knowledge of only the low-energy structure of the theory. As an example, I calculate the leading quantum corrections to the Newtonian gravitational potential.

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The Newtonian potential for the gravitational interactions

$$V(r) = -\frac{Gm_1m_2}{r} \quad (1)$$

is of course only approximately valid. For large masses and/or large velocities there are relativistic corrections which have been calculated within the framework of the general theory of relativity [1], and which have been verified experimentally. At microscopic distance scales, we would also expect that quantum mechanics would lead to a modification in the gravitational potential in much the same way that the radiative corrections of quantum electrodynamics (QED) leads to a modification of the Coulombic interaction [2]. The present paper addresses these quantum corrections to the gravitational interaction.

General relativity forms a very rich and subtle classical theory. However, it has not been possible to combine general relativity with quantum mechanics to form a satisfactory theory of quantum gravity. One of the problems, among others, is that general relativity does not fit the present paradigm for a fundamental theory, that of a renormalizable quantum field theory. Although the gravitational fields may be successfully quantized on smooth-enough background space-times [3], the gravitational interactions are of such a form as to induce divergences which cannot be absorbed by a renormalization of the parameters of the minimal general relativity [3-5]. If one introduces new coupling constants to absorb the divergences, one is led to an infinite number of free parameters. This lack of predictivity is a classic feature of nonrenormalizable field theories. The purpose of this paper is to argue that, despite this situation, the leading long distance quantum corrections are reliably calculated in quantum gravity. The idea is relatively simple and will be the focus of this Letter, with more details given in a subsequent paper [6].

The key ingredient is that the leading quantum corrections at large distance are due to the interactions of massless particles and only involve their coupling at low energy. Both of these features are known from general relativity even if the full theory of quantum gravity is quite different at short distances.

The action of gravity is determined by an invariance

under general coordinate transformations, and will have the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{2}{\kappa^2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu} R^{\nu\alpha} R_{\alpha}^{\mu} + \dots \right]. \quad (2)$$

(We ignore the possibility of a cosmological constant, which experimentally must be very small.) Here  $R$  is the curvature scalar,  $R_{\mu\nu}$  is the Ricci tensor,  $g = \det g_{\mu\nu}$ , and  $g_{\mu\nu}$  is the metric tensor. Experiment determines [1]  $\kappa^2 = 32\pi G$ , where  $G$  is Newton's constant, and [7]  $|\alpha|, |\beta| \leq 10^{74}$ . The minimal general relativity consists of keeping only the first term, but higher powers of  $R$  are not excluded by any known principle. The reason that the bounds on  $\alpha, \beta$  are so poor is that these terms have very little effect at low energies/long distance. The quantities  $R$  and  $R_{\mu\nu}$  involve two derivatives acting on the gravitational field (i.e., the metric  $g_{\mu\nu}$ ). In an interaction each derivative becomes a factor of the momentum transfer involved,  $q$ , or of the inverse distance scale  $q \sim \hbar/r$ . We will say that  $R$  is of order  $q^2$ . In contrast,  $R^2$  or  $R_{\mu\nu} R^{\mu\nu}$  are of order  $q^4$ . Thus, at small enough energies, terms of order  $R^2, R^3$ , etc. are negligible and we automatically reduce to only the minimal theory.

The quantum fluctuations of the gravitational field may be expanded about a smooth background metric [3], which in our case is flat space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\ \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (3)$$

About a decade ago, there was extensive study of the divergences induced in one and two loop diagrams, also including matter fields [3-5,8,9]. When starting from the Einstein action, the divergences appear at higher order, i.e., in  $\alpha, \beta$  for one loop and in  $\gamma$  at two loops. This is not hard to see on dimensional grounds; the expansion is in powers of  $\kappa^2 q^2 \sim Gq^2$  which forms a dimensionless combination. These divergences can be absorbed into renormalized values of the parameters  $\alpha, \beta, \gamma$ , which could in principle be determined by experiment. As mentioned before, higher loops will require yet more arbitrary parameters.

However, also contained in one loop diagrams are fi-

nite corrections of a different character. These are non-analytic contributions, which around flat space have the form  $\kappa^2 q^2 \ln(-q^2)$  or  $\kappa^2 q^2 \sqrt{\frac{m^2}{-q^2}}$ . Because these are non-analytic, e.g., picking up imaginary parts for timelike  $q^2$  ( $q^2 > 0$ ), they cannot be absorbed into a renormalization of parameters in a local Lagrangian. Also, because  $|\ln(-q^2)| \gg 1$  and  $|\sqrt{\frac{m^2}{-q^2}}| \gg 1$  for small enough  $q^2$ , these terms will dominate over  $\kappa^2 q^2$  effects in the limit  $q^2 \rightarrow 0$ . Massive particles in loop diagrams do not produce such terms; a particle with mass will yield a local low energy Lagrangian when it is integrated out of a theory, yielding contributions to the parameters  $\alpha, \beta, \gamma$  of the Lagrangian in Eq. (2). In contrast, nonanalytic contributions come from long distance propagation, which at low energy is only possible for massless particles. Similarly, to determine the coefficients of the long distance nonanalytic terms, one does not have to know the short distance behavior of the theory; only the lowest energy coupling is required. Since both the enumeration of the massless particles and the low energy coupling constant follow from the Einstein action, this is sufficient to determine the dominant low energy corrections.

The above argument is at the heart of the paradigm of effective field theories [10,11], which have been developed increasingly in the past decade. Indeed it is almost identical to the way that low energy calculations involving pions are performed in chiral perturbation theory, which is an effective field theory representing the low energy limit of QCD. [There the role of  $\kappa^2$  is taken by  $1/16\pi^2 F_\pi^2 \approx 1/(1 \text{ GeV})^2$  and the higher order renormalized constants equivalent to  $\alpha, \beta$  are of order  $10^{-3}$ .] The interested reader is directed to the literature of chiral perturbation theory [10–12] to see how an effective field theory works in practice, including comparison with experiment. It has recently been shown that the sicknesses of  $R + R^2$  gravity are not problems when treated as an effective field theory [13].

Let us see how this technique works in the case of the Newtonian potential. When one adds a heavy external source, use of the action of Eq. (2) plus one graviton exchange leads to a classical potential of the form [7]

$$\begin{aligned} V(r) &= \frac{Gm_1 m_2}{r} \left[ 1 - \frac{4}{3} e^{-r/r_2} + \frac{1}{3} e^{-r/r_0} + \dots \right] \\ &= Gm_1 m_2 \left[ \frac{1}{r} - 128\pi^2 G(\alpha + \beta) \delta^3(\mathbf{x}) + \dots \right], \\ r_2^2 &= -16\pi G\beta, \\ r_0^2 &= 32\pi G(3\alpha + \beta). \end{aligned} \quad (4)$$

Simply put, the effect of the order  $q^4$  effects of  $R^2$  and  $R_{\mu\nu}R^{\mu\nu}$  are short ranged. (The second line above indicates that these terms limit to a Dirac delta function as  $\alpha, \beta \rightarrow 0$ . This second form of the potential is most appropriate for a perturbation in an effective field theory.) In contrast the leading quantum corrections will fall like powers of  $r$ , and hence will be dominant at large  $r$ .

In order to calculate the quantum corrections we need to specify the propagators and vertices of the theory. It is most convenient to use the harmonic gauge,  $2\partial_\mu h_\nu^\mu = \partial_\nu h_\lambda^\lambda$ , which is accomplished by including the following gauge fixing term:

$$\mathcal{L}_{\text{gf}} = \sqrt{-g} \left[ D_\sigma h_\mu^\sigma - \frac{1}{2} D_\mu h_\sigma^\sigma \right] g^{\mu\nu} \left[ D_\lambda h_\nu^\lambda - \frac{1}{2} D_\nu h_\lambda^\lambda \right]. \quad (5)$$

The most useful feature of this gauge is the relative simplicity of the graviton propagator, which assumes the form

$$\begin{aligned} D_{\mu\nu, \alpha\beta}(q) &= \frac{i}{q^2} P_{\mu\nu, \alpha\beta}, \\ P_{\mu\nu, \alpha\beta} &= \frac{1}{2} [\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}]. \end{aligned} \quad (6)$$

We will follow the same procedure of calculating radiative corrections as is done for the Coulomb potential in QED. The one loop diagrams are shown in Fig. 1. The coupling to an external graviton field  $h_{\mu\nu}^{\text{ext}}$  involves the energy momentum tensor

$$\mathcal{L}_I = -\frac{\kappa}{2} h_{\mu\nu}^{\text{ext}} T^{\mu\nu}. \quad (7)$$

For an external spinless source with Lagrangian,

$$\mathcal{L}_M = \frac{\sqrt{-g}}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2], \quad (8)$$

the tensor is

$$T_{\mu\nu}^M = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2), \quad (9)$$

while for two gravitons it is longer,

$$\begin{aligned} T_{\mu\nu}^h &= -h^{\sigma\lambda} \partial_\mu \partial_\nu h_{\sigma\lambda} + \frac{1}{2} h \partial_\mu \partial_\nu h \\ &+ \left( \frac{1}{4} \partial_\mu \partial_\nu - \frac{3}{8} \eta_{\mu\nu} \square \right) [hh - 2h^{\sigma\lambda} h_{\sigma\lambda}] \\ &- \square [h_{\sigma\mu} h_\nu^\sigma - hh_{\mu\nu}] - (\partial_\lambda \partial_\mu [hh_\nu^\lambda] + \partial_\lambda \partial_\nu [hh_\mu^\lambda]) \\ &+ 2\partial_\sigma \partial_\lambda [h_\mu^\sigma h_\nu^\lambda - h^{\sigma\lambda} h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h^{\lambda\rho} h_\rho^\sigma + \frac{1}{2} \eta_{\mu\nu} hh^{\sigma\lambda}] \\ &+ 2\partial_\lambda [h^{\lambda\sigma} \partial_\mu h_{\sigma\nu} + h^{\lambda\sigma} \partial_\nu h_{\sigma\mu}] \\ &- (h_\mu^\sigma \square h_{\sigma\nu} + h_\nu^\sigma \square h_{\sigma\mu} - h_{\mu\nu} \square h) \\ &+ \frac{1}{2} \eta_{\mu\nu} [h^{\lambda\sigma} \square h_{\lambda\sigma} - \frac{1}{2} h \square h], \end{aligned} \quad (10)$$

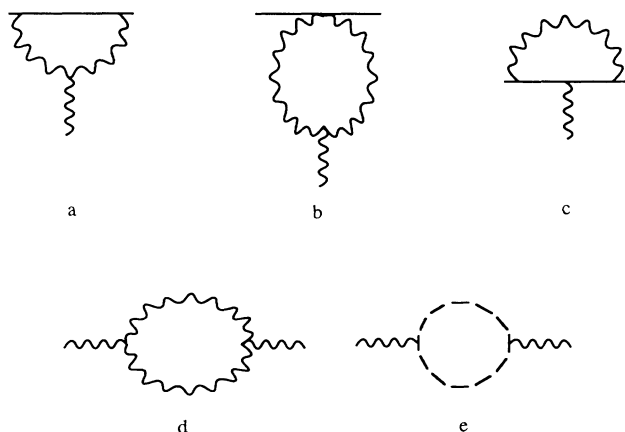


FIG. 1. One loop radiative corrections to the gravitational vertex, (a)–(c), and vacuum polarization, (d),(e).

where  $h = h_\lambda^\lambda$ . The two graviton matter vertex in Fig. 1(b) follows from the Lagrangian

$$\mathcal{L}_2 = +\kappa^2 \left( \frac{1}{2} h^{\mu\lambda} h_\lambda^\nu - \frac{1}{2} h h^{\mu\nu} \right) \partial_\mu \phi \partial_\nu \phi - \frac{1}{8} \kappa^2 (h^{\lambda\sigma} h_{\lambda\sigma} - \frac{1}{2} h h) [\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2]. \quad (11)$$

Gauge fixing is accomplished in path integral quantization by use of Fadeev-Popov ghosts,  $\eta_\mu$ . The ghost Lagrangian is [3]

$$\mathcal{L}_{\text{ghost}} = \sqrt{-g} \eta^{\mu*} [\square \eta_{\mu\nu} - R_{\mu\nu}] \eta^\nu. \quad (12)$$

Collectively, these Lagrangians define the vertices re-

quired to compute Feynman diagrams.

The calculation of the vertex correction is straightforward but algebraically tedious. Figure 1(c) does not lead to any nonanalytic terms because the coupling is to the massive particle. (It does have an infrared divergence like the one in QED, which can be handled in a similar fashion [2].) In general the radiative corrected matrix element will have the form

$$V_{\mu\nu} = \langle p' | T_{\mu\nu} | p \rangle = F_1(q^2) [p'_\mu p_\nu + p_\mu p'_\nu + q^2 \eta_{\mu\nu}] + F_2(q^2) [q_\mu q_\nu - g_{\mu\nu} q^2], \quad (13)$$

with  $F_1(0) = 1$ . For the first two diagrams the nonanalytic terms are found to be

$$\begin{aligned} \Delta F_1 &= \frac{\kappa^2 q^2}{32\pi^2} \left[ -\frac{3}{4} \ln(-q^2) + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}} \right], & \Delta F_2 &= \frac{\kappa^2 m^2}{32\pi^2} \left[ 3 \ln(-q^2) + \frac{7}{8} \frac{\pi^2 m}{\sqrt{-q^2}} \right], & \text{for Fig. 1(a)} \\ \Delta F_1 &= 0, & \Delta F_2 &= \frac{\kappa^2 m^2}{32\pi^2} \left[ -\frac{13}{3} \ln(-q^2) \right], & \text{for Fig. 1(b)} \end{aligned} \quad (14)$$

so that

$$F_1(q^2) = 1 + \frac{\kappa^2 q^2}{32\pi^2} \left[ -\frac{3}{4} \ln(-q^2) + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}} \right] + \dots, \quad F_2(q^2) = \frac{\kappa^2 m^2}{32\pi^2} \left[ -\frac{4}{3} \ln(-q^2) + \frac{7}{8} \frac{\pi^2 m}{\sqrt{-q^2}} \right]. \quad (15)$$

The vacuum polarization diagram has been calculated previously [3]. In dimensional regularization with only massless particles the  $\ln(-q^2)$  terms can be read off from the coefficient of the  $1/(d-4)$  in a one loop graph. This yields the nonanalytic terms

$$P_{\mu\nu, \alpha\beta} \Pi_{\alpha\beta, \gamma\delta} P_{\gamma\delta, \rho\sigma} = \frac{\kappa^2}{32\pi^2} q^4 \left[ \frac{21}{120} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) + \frac{1}{120} \eta_{\mu\nu} \eta_{\rho\sigma} \right] [-\ln(-q^2)], \quad (16)$$

where I have dropped many terms proportional to  $q_\mu, q_\nu$ , etc., which because of gauge invariance do not contribute to the interaction described below.

The most precise statement of the one loop results is in terms of the relativistic forms given above, Eqs. (13) (16). However, it is pedagogically useful to combine these to define a potential. I will define this as the sum of one particle reducible diagrams. For a two body interaction, one obtains this potential from the Fourier transform of the nonrelativistic limit of Fig. 2, where the blobs indicate the radiative corrections. In momentum space we have

$$\begin{aligned} & -\frac{\kappa^2}{4} \frac{1}{2m_1} V_{\mu\nu}^{(1)}(q) [iD_{\mu\nu, \alpha\beta}(p) + iD_{\mu\nu, \rho\sigma} i\Pi_{\rho\sigma, \eta\lambda} iD_{\eta\lambda, \alpha\beta}] V_{\alpha\beta}^{(2)}(q) \frac{1}{2m_2} \\ & \approx 4\pi G m_1 m_2 \left[ \frac{i}{\mathbf{q}^2} - \frac{i\kappa^2}{32\pi^2} \left( -\frac{127}{60} \ln \mathbf{q}^2 + \frac{\pi^2 (m_1 + m_2)}{2\sqrt{\mathbf{q}^2}} \right) \right], \end{aligned} \quad (17)$$

where the second line corresponds to the nonrelativistic limit  $p_\mu = (m, 0), q = (0, \mathbf{q})$ . In taking the Fourier transforms, we use

$$\begin{aligned} \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{1}{\mathbf{q}^2} &= \frac{1}{4\pi r}, \\ \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{1}{q} &= \frac{1}{2\pi^2 r^2}, \\ \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \ln \mathbf{q}^2 &= \frac{-1}{2\pi^2 r^3}. \end{aligned} \quad (18)$$

If we reinsert powers of  $\hbar$  and  $c$  at this stage, we obtain the potential energy

$$V(r) = -\frac{GM_1 M_2}{r} \left[ 1 - \frac{G(M_1 + M_2)}{rc^2} - \frac{127}{30\pi^2} \frac{G\hbar}{r^2 c^3} \right]. \quad (19)$$

The first correction, of order  $GM/rc^2$ , does not contain any power of  $\hbar$ , and is of the same form as various post-Newtonian corrections which we have dropped in taking the nonrelativistic limit [1]. In fact, for a small test particle  $M_2$ , this piece is the same as the expansion of the time component of the Schwarzschild metric,

$$g_{00} = \frac{1 - \frac{GM_1}{rc^2}}{1 + \frac{GM_1}{rc^2}} \approx 1 - \frac{2GM_1}{rc^2} \left[ 1 - \frac{GM_1}{rc^2} \right], \quad (20)$$

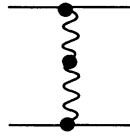


FIG. 2. Diagrams included in the potential. The dots indicate vertices and propagators including the corrections shown in Fig. 1.

which is the origin of the static gravitational potential. Therefore we do not count this result as a quantum correction. However, the last term is a true quantum effect, linear in  $\hbar$ . We note also that if the photon and neutrinos are truly massless, they too must be included in the vacuum polarization diagram. Using the results of Ref. [8], this changes the quantum modification to

$$-\frac{135 + 2N_\nu}{30\pi^2} \frac{G\hbar}{r^2 c^3}, \quad (21)$$

where  $N_\nu$  is the number of massless helicity states of neutrinos.

The effect calculated here is distinct from another finite contribution to the energy momentum vertex—the trace anomaly [14]. The trace anomaly is a local effect and is represented by analytic corrections to the vertices, while the crucial distinction is that the nonanalytic terms are nonlocal. Note that the quantum correction above is *far* too small to be measured. However, the specific number is less important than the knowledge that a prediction can be made.

The ability to make long distance predictions certainly does not solve all of the problems of quantum gravity. Most likely the theory must be greatly modified at short distances, for example, as is done in string theory. Most quantum predictions involving gravity treat quantum matter fields in a classical gravitational field [14]. True predictions (observable in principle and without unknown parameters) involving the quantized gravitational field are few. However, the methodology of effective field theory, when applied to gravity, yields well defined quantum predictions at large distances.

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