Evading Amplifier Noise in Nonlinear Oscillators

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Resonators driven to self-oscillation via active feedback play an important role in technology. Among the stochastic processes driving phase diffusion in such oscillators is noise from the feedback amplifier. Here a technique is described by which phase diffusion due to this noise can be suppressed. We have achieved a 10 dB reduction in phase diffusion by using the technique on an oscillator whose frequency-controlling element is a nonlinear mechanical resonator. The technique, in principle, provides a quantum nondemolition method of tracking a resonator's phase.

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Free running oscillators consisting of resonators driven to self-oscillation via active feedback have considerable technological importance. This importance stems from the fact that these oscillators form the essential elements of continuous-wave frequency sources and clocks and are often the basis of other sensitive and accurate measuring devices [1]. Generally, the amplifier or active gain medium that feeds power back into the resonator to maintain oscillation will also inject noise into the resonator. This noise is a source of phase diffusion or frequency jitter. Here we show that if the resonator has a cubic reactive nonlinearity, amplifier noise can be evaded, making it possible, in principle, to reach the regime where the long term frequency stability of the oscillator is determined solely by the noise associated with the resonator's intrinsic loss.

The presence of a cubic nonlinearity in the restoring force leads to a distortion of the resonance curve (amplitude as a function of frequency) which appears as a pulling of the Lorentzian line shape [2]. This distortion grows progressively with increasing drive level, eventually giving rise to a resonance curve which is multivalued for a range of frequencies. A portion of the resonance curve now corresponds to unstable oscillation, if the resonator is driven without feedback. However, in the scheme considered here, phase information is fed back to the resonator and this leads to stable operation throughout parameter space. This is possible because the frequency remains a single-valued function of the phase. The oscillator can therefore be operated at points along the resonance curve where the slope is infinite. Since the phase-versus-frequency curve is also vertical at these points the frequency is insensitive to the phase of the drive. This leads to the enhancement in the long term phase stability discussed in this Letter.

The technique we propose and demonstrate provides, in principle, a quantum nondemolition method [3,4] for tracking a resonantor's phase [5]. This differs from previous noise squeezing and quantum nondemolition measurement methods applied to mechanical resonators in which only amplitude components were monitored [6,7].

Although the discussion presented here is in the context of oscillators employing mechanical resonators, the technique described is more generally applicable and may even find use in optical frequency sources. It is noted that methods employing nonlinearities and feedback have been proposed and demonstrated for reducing laser linewidth [8,9]. Our method, in contrast, focuses on long term phase stability.

The resonator we consider is a simple unstressed rectangular beam held rigidly at both ends. The beam is placed in a uniform magnetic field oriented perpendicular to the beam's long axis and is driven into vibration in the plane perpendicular to the field via an alternating current flowing through the conducting beam. The amplitude of the beam's motion is sensed by measuring the voltage induced along the conductor.

The linear restoring force associated with small displacements of the beam is due to the bending moments which develop along the beam. Since the beam is anchored at its ends, however, a line tension also develops as the beam is forced from its equilibrium position and its length changes. This force is proportional to the cube of the average displacement Y and is responsible for the nonlinear behavior of the resonator. The equation of motion for the resonator is

$$M\frac{d^2Y}{dt^2}+\mu\frac{dY}{dt}+K_1Y+K_3Y^3=F_D+F_L+F_A\,, \qquad (1)$$

where M is the effective mass of the beam, μ is the damping constant of the loaded resonator, K_1 is the linear spring constant, and K_3 characterizes the strength of the cubic nonlinearity. The force exerted on the resonator by the drive is denoted by F_D; F_L is the fluctuating force associated with the resonator's intrinsic loss (as required by the fluctuation dissipation theorem); and F_A is the force exerted on the resonator by the noise current I_{N1} emitted from the input port of the feedback amplifier. This force is given by

$$F_A=lBI_{N1}\,, \qquad (2)$$

where l is the length of the beam and B is the magnetic induction.

Assuming the amplifier has a sufficiently high input impedance, the output of the amplifier with voltage gain G is

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$$V_{\mathrm{out}}=G(V_R+V_{N2}). \tag{3}$$

Here

$$V_R=lB\frac{dY}{dt} \tag{4}$$

is the voltage induced in the resonantor conductor, and V_{N2} is the fluctuating noise voltage (referred to input) generated in the amplifier. Although it is straightforward to treat more general cases, for simplicity the noise sources I_{N1} and V_{N2} will be taken to be statistically independent. The output voltage is fed through a phase shifter and then into an amplitude limiter [10] whose output drives the mechanical resonator.

To describe the action of the phase shifter and the limiter we write V_{out} in the form

$$V_{\mathrm{out}}(t)=V(t)\cos[\Omega t+\phi(t)], \tag{5}$$

where Ω is the self-oscillation frequency. The phase shifter shifts the phase by a constant amount ϕ_C. The limiter replaces the voltage $V(t)$ with some fixed voltage. The resulting signal drives a current through the resonator conductor providing the force [11]

$$F_D=F_0\cos[\Omega t+\phi(t)+\phi_C]. \tag{6}$$

The two constants ϕ_C and F_0 are set by the experimenter and determine the self-oscillation frequency Ω which usually differs from the natural oscillation frequency $\Omega_0=\sqrt{K_1/M}$. Typically the detuning $\omega=\Omega-\Omega_0$ will be small compared with Ω_0.

In general, the noise sources I_{N1}, V_{N2}, and F_L will cause the phase of the oscillator to diffuse. We now present the results of a straightforward analysis in which one linearizes about the steady state solution to determine the system's response to the three noise sources. It is shown that by the proper choice of operating point, the nonlinearity of the resonator can be exploited to make the long term phase wandering of the oscillator immune to V_{N2}. Further, by making the coupling between the resonator and the amplifier sufficiently weak the effect of I_{N1} on phase diffusion can be made negligible compared to the intrinsic noise F_L. The oscillator's long term phase stability will then be determined solely by the quality of the resonator. Phase stability translates into frequency stability since the two are related through $\delta f=\delta\phi/2\pi\tau$ where $\delta\phi$ denotes the root-mean-square (rms) deviation from the mean phase accumulated in the time interval τ, and δf is the rms deviation of the frequency from its mean value as reported by a frequency counter with counting time τ.

We first consider the case of an oscillator with a *linear* resonator which is operated at the maximum of the resonance curve. At this point this system exhibits optimum performance with respect to phase diffusion. The expression for $(\delta f)^2$ is

$$(\delta f)^2=(D_{N1}+D_{N2}+D_L)/4\pi^2\tau. \tag{7}$$

The constant D_{N1} characterizing the diffusion due to noise from the amplifier's input is given by

$$D_{N1}=\langle V_R^2\rangle i_{\mathrm{rms}}^2/16\mathcal{E}^2, \tag{8}$$

where i_{rms}^2 is the mean square current per unit bandwidth for I_{N1} at the oscillator's resonant frequency and $\mathcal{E}$ is the total energy stored in the resonator. The constant D_{N2} characterizing the diffusion due to noise from the amplifier's output is given by

$$D_{N2}=v_{\mathrm{rms}}^2/4\tau_T^2\langle V_R^2\rangle. \tag{9}$$

Here v_{rms}^2 is the mean square voltage per unit bandwidth for V_{N2} at the oscillator's resonant frequency and $\tau_T=\mu/2M$ is the amplitude ring-down time for the loaded resonator. The constant D_L characterizing diffusion due to noise from the resonator's loss is given by

$$D_L=\eta_L/\tau_T\mathcal{E}. \tag{10}$$

The spectral density η_L for the available noise power from the losses responsible for resonator damping, under thermal equilibrium conditions, is given by $\eta_L=k_BT$ where k_B is Boltzmann's constant and T is the temperature of the resonator.

The expression for $(\delta f)^2$ corresponding to the operation of the oscillator with a nonlinear resonator at the peak of the resonance curve (where $\phi_C=-\pi/2$) is more complicated than Eq. (10). However, in the case when D_{N2} is large compared to both D_{N1} and D_L we have

$$(\delta f)^2=D_{N2}/4\pi^2\tau. \tag{11}$$

Note that D_{N2} does not depend on the strength of the nonlinearity K_3. Therefore, when D_{N2} is the dominant diffusion constant the performance of the nonlinear system operated at the peak of the resonance curve is identical to that of the linear system in which ϕ_C has been optimized to minimize phase diffusion, provided both oscillators have the same amplitude of oscillation. The noise performance of the nonlinear system at the peak of the resonance curve thus provides a convenient reference with which the performance of the optimized nonlinear system can be compared.

We now consider the noise performance at a unique point [12] which is reached for a particular value of the drive F_0 and with ϕ_0 adjusted so that the displacement lags the drive by $2\pi/3$. At this critical point the slope of the resonance curve is infinite but the entire curve remains single valued. The amplitude is $\sqrt{3}/2$ of its maximum value, and the detuning [13] ω is $\sqrt{3}/\tau_T$. At this point,

$$(\delta f)^2=\frac{1}{\pi^2\tau}(D_{N1}+D_L)+\frac{\tau_T}{4\pi^2\tau^2}(D_{N2}-3D_{N1}-3D_L)(1-e^{-\tau/\tau_T}). \tag{12}$$

For $\tau\ll\tau_T$, this equation reduces to Eq. (7) so there is no improvement in the short term phase stability. For τ sufficiently large to satisfy the conditions $\tau\gg\tau_T$ and

$$\tau\gg\frac{D_{N2}}{D_{N1}+D_L}\tau_T, \tag{13}$$

Eq. (12) reduces to

$$(\delta f)^2=(D_{N1}+D_L)/\pi^2\tau . \quad (14)$$

Note the absence of a term containing D_{N2}; at the critical point the amplifier's output noise plays no role in determining the oscillator's long term phase stability. Moreover, since D_{N1} is proportional to the square of the magnetic field via its dependence on V_R, D_{N1} can be made small compared to D_L by operating in a sufficiently small field. Thus by operating the oscillator at the critical point and making the coupling between the resonator and the amplifier sufficiently weak one can make the long term stability of the oscillator independent of amplifier noise. Comparing this result with Eq. (7) one sees that the long term phase stability achieved by operating at the critical point with a noisy amplifier is only a factor of 2 worse than what one could have achieved using a linear resonator and a noiseless amplifier.

The resonator that we employed to demonstrate the feasibility of this technique was etched from a high purity silicon wafer which also provided the support for the beam. The length, width, and thickness of the beam were, respectively, 3.6 mm, 127 μm, and 26 μm. A thin layer of gold evaporated onto one surface provided a conduction path with a low temperature resistance of 17 Ω. The resonator was located in a uniform field of 1 T and mounted on a platform whose temperature was precisely regulated at 100 mK. The resonant frequency (lowest mode) was 15.994 kHz. The quality factor Q was 375000.

Figure 1 shows the phase difference between the drive force and the displacement of the oscillator measured for three values of the drive current. These results were extracted from simultaneous measurements of the in-phase and quadrature components of the amplitude of the response. The solid portions of the curve were mapped out during slow ramps of the frequency. The individual points were obtained with the oscillator operating in the self-excited loop discussed above by making incremental changes in the setting of the phase shifter.

The critical drive was most accurately determined by slowly modulating the phase about $-2\pi/3$ at various drive levels. The accompanying swing in the oscillator's frequency reached its minimum amplitude at a drive of 7.2 nA.

For the studies of the noise behavior of the oscillator the voltage induced along the vibrating beam was fed into two amplifiers. The first formed part of the oscillator loop. Its output, however, was now combined with that of a noise generator. With this modification the effective output noise could be increased so that the resulting frequency jitter of the oscillator was above the noise floor of the frequency counter. The second amplifier led directly to the frequency counter. By this method, the direct broad-band "amplifier" noise was conveniently separated from the narrow-band frequency fluctuations returned by the resonator. This separation also could have been achieved in a single amplifier setup by inserting a very narrow band-pass tracking filter in front of the frequency counter.

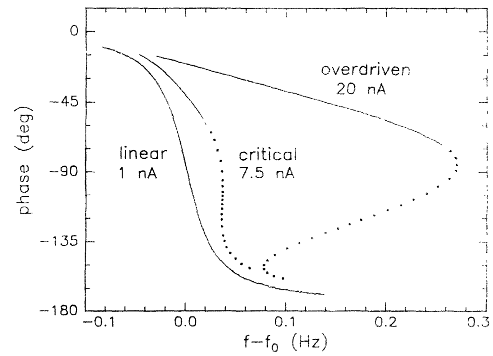

FIG. 1. Measured phase difference between the drive force and the displacement of the nonlinear oscillator plotted as a function of the frequency. Results are shown for three different values of the drive current. The natural resonant frequency f_0 is 15.994 kHz.

Figure 2 shows rms frequency fluctuations measured with an averaging time of 20 sec plotted as a function of the oscillator drive scaled by the inverse of the critical point drive. The solid circles correspond to measurements at the peaks of the resonance curves, i.e., with the phase set at $-90°$. The open circles are data obtained at a phase setting of $-120°$. When the reduced drive is unity at this phase shift the resonator is at the critical point. There is a dramatic difference in the drive dependence for the two sets of data, with the open-circle results exhibiting a well defined minimum centered near a reduced drive of 1. This is as one might have anticipated based on the phase-versus-frequency curves of Fig. 1. Note also that in the vicinity of the minimum the fluctuations fall below those of the linear system (i.e., the solid circles) even though the amplitude of the response is now 15%

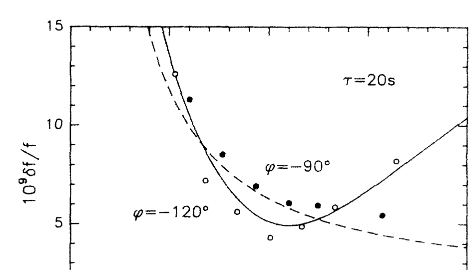

FIG. 2. Frequency fluctuations versus oscillator drive at an averaging time of 20 sec. The solid cricles show results obtained with the oscillator operating at the peaks of the resonance curves where the phase is $-90°$. The open circles were measured at amplitudes equal to $\sqrt{3}/2$ of the peak values where the phase is $-120°$.

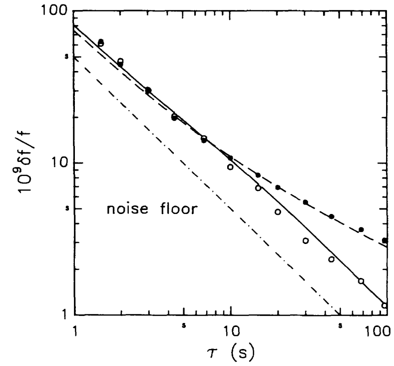

FIG. 3. Frequency fluctuations at the critical drive as a function of counter averaging time. The open circles are data obtained at the critical point. The solid circles are results obtained at the peak of the resonance curve. Note the 10 dB reduction in noise achieved at the critical point for $\tau = 100$ sec.

smaller. The dashed curve is a fit of the resonance peak data to Eq. (11) with corrections applied for the instrument noise floor and for a contribution due to the direct feedthrough of the drive voltage. This latter term, which is proportional to $v_{\rm rms}/V_R\tau$ will be discussed in a subsequent paper along with the theoretical expressions leading to the solid curve describing the data at $-120°$.

The relative improvement in noise performance achieved by operating the oscillator at its critical point becomes greater with increasing counter times as shown in Fig. 3. Here frequency fluctuations measured at the critical drive are plotted as a function of the averaging time on log-log scales. The solid circles were obtained with the phase adjusted so that the oscillator was operating at the peak of the resonance curve. The open circles were obtained at the critical point. For $\tau \lesssim 10$ sec both sets of fluctuations are dominated by the contribution due to the direct pickup of the drive noise and show a $1/\tau$ dependence. For longer times the solid circles show the $1/\sqrt{\tau}$ dependence expected for a linear system. The open circles, however, continue to exhibit a $1/\tau$ dependence out to our longest averaging times. At $\tau = 100$ sec the critical point frequency fluctuations have become a factor of 3 smaller than those of the linear oscillator. Equation (12) predicts that the open-circle data should continue falling with this time dependence until hitting the intrinsic loss curve. The critical point fluctuation data would then cross over to a $1/\sqrt{\tau}$ dependence but now with the amplitude corresponding to the intrinsic loss. The dashed and solid curves in Fig. 3 are fits by Eqs. (11) and (12), respectively, again with D_{N2} assumed large compared to D_{N1} and D_L and with corrections applied both for the instrument noise floor and the feedthrough of the drive signal. The fitted value of $v_{\rm rms}$ is 5.0 V Hz$^{-1/2}$, only 8% larger than the directly measured value. There is thus quantitative agreement between experiment and theory and a direct demonstration that the phase diffusion due to amplifier noise can be significantly reduced by operating at the critical point of a nonlinear oscillator.

Interpreted as Heisenberg equations of motion Eqs. (1) through (6) hold quantum mechanically [14]. The rest of the equations then follow from a straightforward quantum analysis in which quantum mechanics places fundamental lower bounds on the size of the noise emitted by the amplifier and the losses. The technique described here thus allows one to evade amplifier noise, with the consequence that the phase diffusion is determined solely by the quantum noise associated with the resonator loss. If the system is used as a receiver and the resonator loss results from coupling to a signal source the phase diffusion, provided the signal is suitably squeezed, will be much smaller than that corresponding to a signal source consisting of vacuum fluctuations. It is thus argued that the measurement technique described here functions as a quantum-nondemolition measurement method for continuously monitoring the phase of a resonator.

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[10] A National Semiconductor LN319 fast comparator was employed as the limiter. Although the output consists of a square wave whose zero crossings jitter the resonator, because of the high Q, only responds at the fundamental frequency.

[11] Limiter noise was neglected in writing this equation. Limiter noise can be neglected provided the gain of the amplifier is sufficiently high so that the rms fluctuations in the amplifier's output are much larger than the limiter noise referred to the limiter's input and provided that the limiter output is sufficiently weakly coupled to the resonator so that resonator loss noise dominates.

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