Stochastic Resonance in Bistable Systems Driven by Harmonic Noise

Alexander Neiman
Arbeitsgruppe Nichtlineare Dynamik der Max-Planck-Gessellschaft an der Universität Potsdam, PF 601553, D-14415 Potsdam, Germany and Department of Physics, Saratov State University, Astrakhanskaya St. 83, 410071, Saratov, Russia

Lutz Schimansky-Geier*
Institut für Theoretische Physik, Humboldt Universität Berlin, D-10099, Berlin, Germany
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We study stochastic resonance in a bistable system which is excited simultaneously by white and harmonic noise which we understand as the signal. In our case the spectral line of the signal has a finite width as it occurs in many real situations. Using techniques of cumulant analysis as well as computer simulations we find that the effect of stochastic resonance is preserved in the case of harmonic noise excitation. Moreover, we show that the width of the spectral line of the signal at the output can be decreased via stochastic resonance. The last could be of importance in the practical use of the stochastic resonance.

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The main topic in noisy driven nonlinear systems in the last decade is without doubt stochastic resonance (SR) [1]. It is now well known that a small harmonic or polyharmonic signal forcing a stochastic nonlinear system can be amplified via SR [2]. It happens most effectively for an optimal noise intensity of the stochastic force in or applied to the nonlinear system. If the noise controlled time scale (say the transition time between the two stable states of a bistable system or the mean time to reach the boundary of inducing stochastically a certain event) coincides with the time scale of the signal the noise is able to amplify the signal by several magnitudes.

Besides a lot of theoretical work, numerical, and analog simulations [1,3,4] several experimental verifications and results in periodically driven systems were reported. This concerns results in laser [5], in paramagnetic systems [6], or ion transport through channels of cell membranes [7]. Of high interest are investigations about periodically stimulated sensory neuron activity [8] and about the amplification of the information transfer in crayfish receptors showing the typical behavior of SR [9].

SR was investigated in the past for periodically driven systems. However, real signals always have a finite spectral width. Effects of a finite spectral width are, therefore, of great importance, especially for real physical applications. The purpose of this Letter is to study the effects of the finite width of the input signal. We base this on a rather simple model, the stochastic bistable system driven by white noise as the source of the amplification and driven by harmonic noise as the signal. The influence of harmonic noise $y(t)$ on bistable systems was studied in [10,11]. SR was considered in a bistable system with a periodical signal and using the harmonic noise as the source for the amplification [12].

Harmonic noise $y(t)$ was introduced [10] by the two-dimensional stochastic differential equations (SDE),

$$\dot{y}=s,\quad \dot{s}=-\Gamma s-\Omega_0^2 y+\sqrt{2\epsilon\Gamma}\,\eta(t)\,, \tag{1}$$

where $\eta(t)$ is Gaussian white noise with $\langle\eta(t)\rangle=0$, $\langle\eta(t)\eta(t')\rangle=\delta(t-t')$. Equation (1) determines the two-dimensional Ornstein-Uhlenbeck process $y(t),s(t)$ with the power spectrum

$$S_{yy}(\omega)=\frac{\epsilon\Gamma}{\omega^2\Gamma^2+(\omega^2-\Omega_0^2)^2} \tag{2}$$

and the mean square displacements $\langle y^2\rangle=\epsilon/\Omega_0^2$, $\langle s^2\rangle=\epsilon$, $\langle ys\rangle=0$. If $\Omega_0^2>\Gamma^2/4$ then the power spectrum (2) has a peak at the frequency $\omega_p=\sqrt{\Omega_0^2-\Gamma^2/2}$ with the width

$$\Delta\Omega_{\rm in}=\sqrt{\omega_p^2+\Gamma\omega_1}-\sqrt{\omega_p^2-\Gamma\omega_1}\,, \tag{3}$$

where $\omega_1=\sqrt{\Omega_0^2-\Gamma^2/4}$.

Let us consider a simple overdamped bistable system driven by white and harmonic noises. It yields

$$\dot{x}=x-x^3+\sqrt{2D}\,\xi(t)+y(t)\,, \tag{4}$$

where $y(t)$ is harmonic noise from SDE (1) and $\xi(t)$ is Gaussian white noise which is statistically independent from $\eta(t)$.

There are a few theoretical and experimental approaches to investigate SR [2–4]. Since in our case the three-dimensional stochastic process (4) is a stationary one, we cannot simply apply the techniques developed for SR. Nevertheless, SR needs a dynamical description which implies in our case the determination of the power spectrum of the output process $x(t)$. We propose a kind of linear response theory [13] which is based on a cumulant expansion [14].

The Fokker-Planck equation (FPE) for the three-dimensional probability density $p=p(x,y,z,t)$ accordingly to the SDE (4) reads

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$$\partial_t p = -\partial_x(x - x^3 + y)p - \partial_y sp - \partial_s(-\Gamma s - \Omega_0^2 y)p + D\,\partial_{xx}p + \epsilon\Gamma\,\partial_{ss}p \,. \quad (5)$$

The notions for the first and second order cumulants of the process $\{x(t),y(t),s(t)\}$ are

$$\kappa_1(t) \equiv \langle x\rangle, \quad \kappa_2(t) \equiv \langle x^2\rangle - \langle x\rangle^2 ,$$
$$\kappa_{xy}(t) \equiv \langle xy\rangle - \langle x\rangle\langle y\rangle , \quad (6)$$
$$\kappa_{xs}(t) \equiv \langle xs\rangle - \langle x\rangle\langle s\rangle .$$

From the FPE (5) we derive the equations for the evolution of the cumulants (6) and determine their stationary states. It turns out that the stationary values of the first order cumulants are equal to zero and therefore not of interest. We further consider the case of a weak signal ($\epsilon \ll 1$). Then applying ideas of the linear response theory we assume the second cumulant in the form

$$\kappa_2(t) = M_2 + \tilde{\kappa}_2(t) . \quad (7)$$

M_2 denotes the second cumulant of the unperturbated stochastic bistable systems in the absence of harmonic noise ($\epsilon = 0$) [4],

$$M_2 = \left(\frac{D}{2}\right)^{1/2} \frac{D_{-3/2}(-1/\sqrt{2D})}{D_{-1/2}(-1/\sqrt{2D})} , \quad (8)$$

where $D_c(x)$ is the parabolic cylinder function and $\tilde{\kappa}_2(t)$ represents small deviations of M_2 due to the harmonic noise. Then we substitute Eq. (7) into the cumulant equations and neglect the terms of higher order in ϵ. However, because of the nonlinearity of the system, the chain of cumulant equations is unclosed. To close the chain of cumulant equations we use a Gaussian approximation which takes into account first and second order cumulants [14]. We emphasize that this approximation concerns only the deviations induced by the harmonic noise. The non-Gaussian behavior of the fluctuations of the purely white noise driven bistable system are contained in expression (8). Thus, we expect qualitative right results in the case where the strength of the harmonic noise is small, i.e., for small signals only. This will be proven later by the coincidence of the theoretical results with the performed computer simulation for small signals.

The resulting cumulant equations in the Gaussian approximation read

$$\tfrac{1}{2}\dot{\tilde{\kappa}}_2 = \tilde{\kappa}_2 - 6M_2\tilde{\kappa}_2 + \kappa_{xy} ,$$
$$\tfrac{1}{2}\dot{\kappa}_{xy} = \kappa_{xy}(1 - 3M_2) + \langle y^2\rangle + \kappa_{xs} , \quad (9)$$
$$\tfrac{1}{2}\dot{\kappa}_{xs} = \kappa_{xs}(1 - 3M_2 - \Gamma) - \Omega_0^2\kappa_{xy} .$$

From (9) we find the stationary values of the cumulants:

$$\tilde{\kappa}_2 = \frac{\kappa_{xy}}{6M_2 - 1} ,$$
$$\kappa_{xy} = \frac{\epsilon(1 - 3M_2 - \Gamma)}{\Omega_0^2[(1 - 3M_2)(\Gamma - 1 + 3M_2) - \Omega_0^2]} , \quad (10)$$
$$\kappa_{xs} = \frac{\epsilon}{(1 - 3M_2)(\Gamma - 1 + 3M_2) - \Omega_0^2} .$$

In the same way we may derive the equations for the correlation functions,

$$R_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle ,$$
$$R_{xy}(\tau) = \langle x(t)y(t+\tau)\rangle ,$$
$$R_{xs}(\tau) = \langle x(t)s(t+\tau)\rangle .$$

Again for $R_{xx}(\tau)$ we use the linear approximation with respect to the harmonic noise perturbations: $R_{xx}(\tau) = R_0(\tau) + \tilde{R}_{xx}(\tau)$. $R_0(\tau)$ is the correlation function of the unperturbated bistable system with white noise. For the $R_0(\tau)$ we use here the approximation which takes into account only transitions between potential wells [4]:

$$R_0(\tau) = M_2 \exp(-\gamma\tau), \quad \gamma = \frac{\sqrt{2}}{\pi}\exp\left(-\frac{1}{4D}\right) . \quad (11)$$

Then again in Gaussian approximation we obtain the equations

$$\frac{d\tilde{R}_{xx}}{d\tau} = (1 - 3M_2)\tilde{R}_{xx} - 3\tilde{\kappa}_2 R_0(\tau) + R_{xy}(\tau) ,$$
$$\frac{dR_{xy}}{d\tau} = R_{xs}, \quad \frac{dR_{xs}}{d\tau} = -\Gamma R_{xs} - \Omega_0^2 R_{xy} , \quad (12)$$

with the initial conditions

$$\tilde{R}_{xx}(0) = \tilde{\kappa}_2, \quad R_{xy}(0) = \kappa_{xy}, \quad R_{xs}(0) = \kappa_{xs} . \quad (13)$$

The power spectrum $S(\omega)$ then follows as $S(\omega) = S_0(\omega) + \tilde{S}(\omega)$, where $S_0(\omega)$ corresponds to the $R_0(\tau)$ and $\tilde{S}(\omega)$ is the Fourier transformation of $\tilde{R}_{xx}(\tau)$. It yields the final expression for the power spectrum,

$$S(\omega) = \frac{\gamma M_2}{\gamma^2 + \omega^2}\left(1 + \frac{3C\tilde{\kappa}_2}{\alpha + \gamma}\right) + \frac{2\epsilon\Gamma C}{(\alpha^2 + \Omega_0^2)^2 - \alpha^2\Gamma^2}\,\frac{\alpha^2 + 2\omega_p^2 - \omega^2}{(\Omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2} , \quad (14)$$

where $\alpha = 1 - 3M_2$ and the constant C is defined from the initial conditions (13).

The power spectrum has a peak at the frequency $\omega_{\max}$, which can be determined from Eq. (14). The frequency depends on the parameters of the system due to the existence of correlations between processes $x(t)$ and $y(t)$. The dependence of $\omega_{\max}$ on the intensity of white noise is shown in Fig. 1. The dotted line corresponds to the frequency ω_p at which the spectrum of the input signal takes its maximum. It is interesting that there exists the value of noise intensity ($D \approx 0.23$) at which the shift between

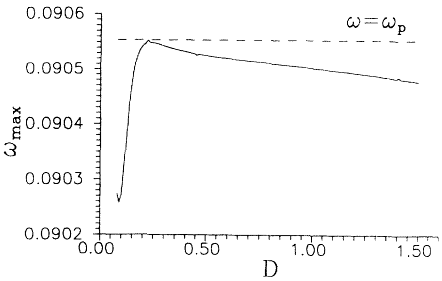

FIG. 1. The dependence of the central frequency of the output signal vs white noise intensity. The dotted line corresponds to the central frequency of the input signal. The parameters are $\Gamma=0.06$, $\Omega_0=0.1$, $\epsilon=0.025$.

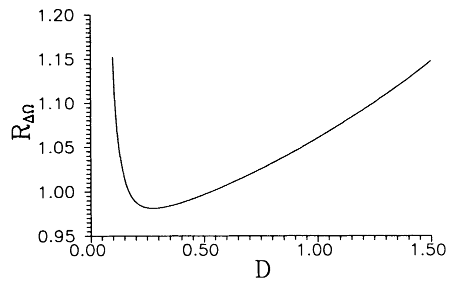

FIG. 3. The dependence of $R_{\Delta\Omega}$ vs D for the same parameters as in Fig. 1.

ω_{max} and ω_p takes its extremum and is equal to zero. Note that in the case of a periodical signal such a dependence cannot be observed, since the system is nonautonomous, the Fokker-Planck equation has a periodical solution, and as a consequence the power spectrum always contains the δ peak at the signal frequency.

Now we consider the signal-to-noise ratio (SNR). We define the SNR (in dB) as

$$R=10\log\frac{S(\omega_{max})}{S_0(\omega_{max})}\,. \tag{15}$$

The dependence of the SNR versus the white noise intensity D is shown in Fig. 2. The SNR takes its maximum at the noise intensity $D\approx 0.46$. Thus, the phenomenon of SR is preserved in the case of harmonic noise.

From the practical point of view of possible applications it is interesting to estimate the width of the spectral peak of the signal at the output. The width of the spectral peak can be obtained from Eq. (14) for the output power spectrum. To estimate the changes of the signal spectral width we introduce the ratio

$$R_{\Delta\Omega}=\frac{\Delta\Omega_{out}}{\Delta\Omega_{in}}\,, \tag{16}$$

where $\Delta\Omega_{in}$ is the signal spectral width at the input (3) and $\Delta\Omega_{out}$ is the signal spectral width at the output. The dependence of $R_{\Delta\Omega}$ versus the white noise intensity D is shown in Fig. 3. It is seen that the case of maximal amplification corresponds to the minimum in dependence on $R_{\Delta\Omega}(D)$. Hence, the spectral width of the signal at the output can be decreased via stochastic resonance.

The algorithm for simulation of a bistable system driven by harmonic noise was developed in [10]. The main advantage of this method is that it allows the solution of the SDE (1) without the procedure of integration. We simulated the SDE (4) and calculated the power spectrum using fast Fourier transform.

Results of simulations of the power spectrum are shown in Fig. 4 for fixed parameters ($\Omega_0=0.1$, $\epsilon=0.025$, $D=0.4$) and for $\Gamma=0.06$, 0.001. For small values of Γ, peaks at the odd harmonics of ω_{max} appear which results from the nonlinearity of the system. Results of the computation of the SNR are presented in Fig. 5 and show qualitative agreement with cumulant analysis. Thus, we also have shown numerically the existence of Sr in the case of excitation by harmonic noise. The new topics in our theory, the shift of the frequency as well as the decrease of the width of the output signal in the resonant case, need further experimental verification.

We have studied a bistable overdamped system driven simultaneously by white and harmonic noise. The latter one can be considered as an appropriate model of real signals which always have finite spectral widths. We found theoretically and numerically the effect of SR. For a cer-

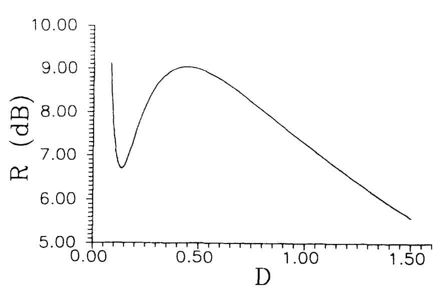

FIG. 2. SNR vs D for the same parameters as in Fig. 1.

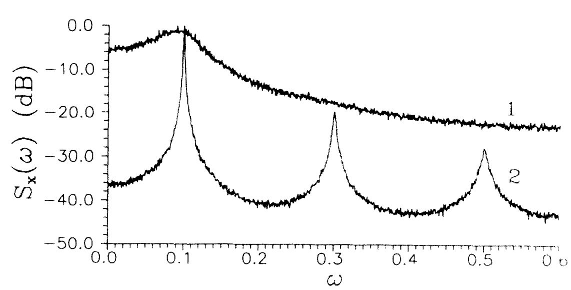

FIG. 4. Numerically simulated power spectrum for the parameters $\Omega_0=0.1$, $\epsilon=0.025$: (1) for $\Gamma=0.6$ and (2) for $\Gamma=0.001$.

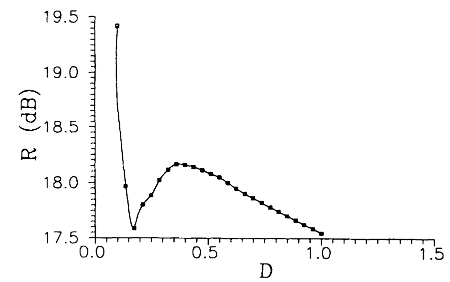

FIG. 5. SNR vs white noise intensity from simulations of SDE (5). The parameters are the same as in Fig. 2.

tain intensity of white noise the signal (harmonic noise) is amplified. It results in an increase of the SNR.

The driving of a bistable system by a signal with a finite width around some central frequency raises new questions. Several topics can only be investigated due to the continuous spectrum of the signal. Thus we found theoretically a decrease of the spectral width of the output signal. Also, the frequency at which the output spectrum takes its maximum depends on the noise intensity. Maximal amplification is reached in the SR case at the central frequency of the input signal which we understand as a synchronization of the bistable system.

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*To whom correspondence should be addressed.

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