Scale Invariant Mixing Rates of Hydrodynamically Unstable Interfaces

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The late time evolution and structure of 2D Rayleigh-Taylor and Richtmyer-Meshkov bubble fronts is calculated, using a new statistical merger model based on the potential flow equations. The merger model dynamics are shown to reach a scale invariant regime. It is found that the Rayleigh-Taylor front reaches a constant acceleration, growing as $0.05gt^2$, while the Richtmyer-Meshkov front grows as $at^{0.4}$ where a depends on the initial perturbation. The model results are in good agreement with experiments and simulations.

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The Rayleigh-Taylor (RT) instability occurs at an interface between a heavy fluid supported by a lighter fluid in a gravitational field [1]. The Richtmyer-Meshkov (RM) instability occurs when a shock wave passes an interface between two fluids of different densities [2]. Perturbations of the unstable interfaces grow and develop, at late times, into turbulent mixing zones. The evolution and structure of these mixing zones is a subject of ongoing research in many fields, such as astrophysics and inertial confinement fusion [3–6].

In recent years several models of the RT mixing front evolution have been suggested [1,4,6–9]. Many of these models are based on the observation that the mixing front is topped by column-shaped bubbles of light fluid, rising and competing [7–10]. At the late nonlinear stage, large bubbles rise faster than smaller ones. A bubble adjacent to smaller bubbles expands and accelerates while its smaller neighbors shrink and are washed downstream. This process leads to a constant growth of the surviving bubbles and to an acceleration of the front. This description of the mixing front was pioneered by Sharp and Wheeler (SW), who proposed a model for bubble rise and merger [1]. Numerical studies of the SW model [7] show that a constant front acceleration is attained. Studies of a simplified version of the SW model by Glimm *et al.* [8], based on an assumption of superposition between single bubble and the local front velocities, showed that the dynamics of the model flow to a fixed point. This corresponds to a constant acceleration which was found to be in agreement with experimental data [8]. In these studies, however, the bubble merger rate is based on a superposition principle and not directly derived from the hydrodynamic equations. A different approach was taken by Zufiraia [9], who constructed a model in which the bubbles are described by potential flow point sources in a uniform flow field. Numerical solution of the equations for a few dozen bubbles yields an acceleration for the top few bubbles in accord with experimental results. However, the flow potential that was used does not give the correct single bubble linear and intermediate growth rates, and thus should not be expected to yield quantitatively correct results in the nonlinear interaction and competition stages. It is also impractical to solve the model for large bubble ensembles necessary for statistical analysis of the front structure and evolution. The late time behavior of the RM mixing front has been studied mainly by experiments and direct simulations [5]. There are, to our knowledge, no published models of RM bubble competition.

In this paper we calculate the mixing rates and structure size distributions of both the RT and single-shock RM bubble fronts between incompressible fluids with a high density ratio (Atwood number $A=1$) in two dimensions. For this purpose we combine a new statistical merger model, first presented in Ref. [11], with a new hydrodynamic model of bubble competition, which is an extension of Layzer's single-bubble model [10]. This hydrodynamic model is used to calculate the single-bubble evolution and two-bubble merger rates. Thus the front evolution and scale invariant bubble size distributions are derived with no free parameters. The model dynamics reach a scale invariant regime, explaining the observed independence of the mixing rates on the initial conditions.

We begin by describing a bubble merger model, in the spirit of the Sharp-Wheeler model [1,7], which allows the use of realistic merger rates and treatment of various types of flow problems. In this model the bubbles are arranged along a line, and are characterized by their wavelengths (diameters) λ_i. Each bubble rises with a velocity $u(\lambda_i)$ equal to the asymptotic velocity of a periodic array of bubbles with wavelength λ_i. Two adjacent bubbles of diameters λ_i and λ_{i+1} merge at a rate $\omega(\lambda_i,\lambda_{i+1},t)$, forming a new bubble of wavelength $\lambda_i+\lambda_{i+1}$. This represents the surviving bubble expanding to fill the space vacated by the bubble washed away from the front. The interface height, $h(t)$, is found by using the average bubble velocity $dh(t)/dt=\langle u\rangle$. We define the size distribution function $g(\lambda,t)d\lambda$ as the number of bubbles with wavelength within $d\lambda$ of λ. The evolution of $g(\lambda,t)$, neglecting near-neighbor correlations, is given by

$$N(t)\frac{\partial g(\lambda,t)}{\partial t}=-2g(\lambda,t)\int_0^\infty g(\lambda',t)\omega(\lambda,\lambda')d\lambda' +\int_0^\lambda g(\lambda-\lambda',t)g(\lambda',t)\omega(\lambda-\lambda',\lambda')d\lambda', \quad (1)$$

where $N(t)=\int_0^\infty g(\lambda,t)d\lambda$ is the number of bubbles at time t. The first term on the right-hand side of Eq. (1) is the rate of elimination of bubbles of wavelength λ by mergers with other bubbles, and the second term is the rate of creation of bubbles of wavelength λ by the merger of two smaller bubbles. We note that numerical simulations of the model and the solution found by numerically evaluating Eq. (1), which neglects correlations in the model, are nearly identical, because negligible neighbor correlations are produced for most merger rates. Analyzing the model [11] it is found that for a large class of merger rates the dynamics reach a scale invariant regime, where the size distribution function scales with the average bubble size.

In order to quantitatively derive the front evolution and size distribution, the single-bubble motion and two-bubble merger rate must be supplied. These are calculated in the present work using a simple potential flow model of bubble evolution at an interface between an incompressible fluid and a constant supporting pressure (or, equivalently, a much lighter fluid). This model is an extension of a single-bubble flow model by Layzer [10]. We consider two bubbles rising in a two dimensional container or vertical channel (or, equivalently, a periodic array of two bubbles of different sizes). The gravitational acceleration is $-g\hat{\mathbf{z}}$, and the initial interface is along the $\hat{\mathbf{x}}$ axis. The bubble tip coordinates are $(x_i,z_{0i}(t))$, $i=1,2$, with $x_1=0$ and $x_2=L$. The flow potential is assumed to be a sum of simple harmonic functions with the problem symmetry and time dependent parameters:

$$\phi(z,x,t)=\sum_{n=1}^{N}a_n(t)\phi_n(x,z)\,, \qquad \phi_n(x,z)=\cos(nkx)e^{-nkz}\,, \quad (2)$$

where $k=2\pi/L$. The fluid velocity is given by $\mathbf{u}=\nabla\phi$. The flow is assumed to be governed by the behavior in the vicinity of the bubble tips [9,10,12]. Near the tips, the interface is, to second order, $z(x,t)=z_{0i}+z_{1i}(x-x_i)^2$, where z_{0i} are the bubble's heights and z_{1i} are their curvatures. The interface moves with the fluid, as expressed by the kinematic equation [10] $u_z=\partial z/\partial t=u_x\partial z/\partial x$, evaluated at the interface. The dynamics are given by Bernoulli's equation $\partial\phi/\partial t+1/2(u_x^2+u_z^2)+gz=\text{const}$. We expand these equations to second order in $(x-x_i)$ near the bubble tips. The first order terms are automatically satisfied due to symmetry, and the two zero order Bernoulli's equations are equated due to the constant pressure in the supporting fluid. Using the potential of Eq. (2) with $N=3$, this yields seven coupled ordinary differential equations for seven variables: the two bubbles' heights $z_{0i}(t)$, their curvatures $z_{1i}(t)$, and the three potential amplitudes $a_n(t)$. These equations are

$$dz_{0i}/dt=v_i^{(0)},\quad dz_{1i}/dt=v_i^{(2)}-2z_{1i}u_i^{(1)}\,,$$

$$\sum_{n=1}^{N}(\phi_{n1}^{(0)}-\phi_{n2}^{(0)})da_n/dt+\frac{(v_1^{(0)})^2-(v_2^{(0)})^2}{2}+g(z_{01}-z_{02})=0\,, \quad (3)$$

$$\sum_{n=1}^{N}\phi_{ni}^{(2)}da_n/dt+(u_i^{(1)})^2/2+v_i^{(0)}v_i^{(2)}+gz_{1i}=0\,,$$

where $\phi_{ni}^{(0)}=\phi_n(p_i)$, $\phi_{ni}^{(2)}=(\partial_x^2/2+z_{1i}\partial_z)\phi_n(p_i)$, $u_i^{(1)}=\partial_x^2\phi(p_i)$, $v_i^{(0)}=\partial_z\phi(p_i)$, $v_i^{(2)}=(\partial_x^2\partial_z/2+z_{1i}\partial_z^2)\phi(p_i)$, $p_1=(0,z_{01})$, $p_2=(L,z_{02})$, and $i=1,2$. For the single-bubble RT problem the model equations (using equal initial parameters for both bubbles) are identical to Layzer's model and yield the correct linear growth rate and terminal velocity. For initial conditions corresponding to two unequal bubbles, the model shows bubble competition. The model is compared against numerical simulations and is applied to other problems, such as flow in finite fluid layers, in Ref. [13]. The agreement of the model predictions with the simulations was found to be very good.

We now use the bubble merger model to study two dimensional RT and RM bubble fronts. We assume that the light fluid density is much smaller than the heavy fluid density. Thus the potential flow model may be applied to calculate the bubble evolution and interaction. For the RM instability, we model the shock by an impulsive acceleration, and assume that the fluid may be treated as incompressible after the shock has passed [14]. We thus use $g=0$ and initial conditions in velocity (corresponding to the initial velocity imparted by the shock) for this case [15]. The single bubble rise velocity (the asymptotic velocity in a periodic array of bubbles of wavelength λ) can be found analytically [10,16]. The result for the RT case is $u=(6\pi)^{-1/2}\sqrt{g\lambda}$, and for RM bubbles is $u=1/(3\pi)\lambda t^{-1}$ where t is the time since the shock passage. We calculated the merger rate by solving Eq. (3) with many sets of initial perturbations. The following results do not depend on the initial perturbations used. A typical plot of the bubble velocities for the RT case is shown in Fig. 1 (similar behavior is found for the RM bubble velocities multiplied by time t). The bubble velocities initially grow exponentially. Then, the velocities begin to saturate as the bubbles coexist. The competition begins at time t_1, when the smaller bubble velocity is at a maximum. The large bubble expands and accelerates and the smaller bubble decelerates and is washed downstream [17]. The large bubble finally attains the velocity of a bubble of the full container width [18]. We define the end of the merger process at the time t_2, when the smaller bubble velocity reaches zero. At this time the small bubble is considered to be eliminated from

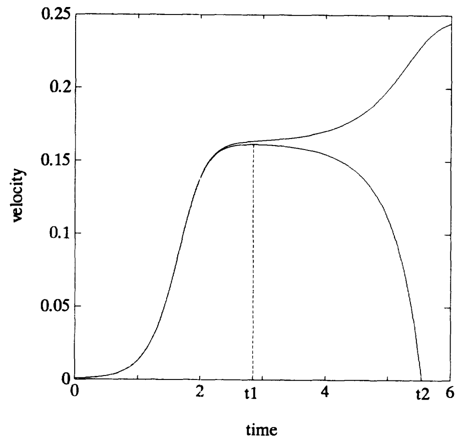

FIG. 1. Bubble velocities for two bubbles in a channel of length $L=1$ with $g=1$. The initial conditions for the flow potential parameters are $a_1=1.27e-6$, $a_2=-6.35e-5$, and $a_3=0$. The exponential rise, coexistence, and competition stages are clearly seen. For this bubble pair, $q=1.03$ and $\omega=0.37$.

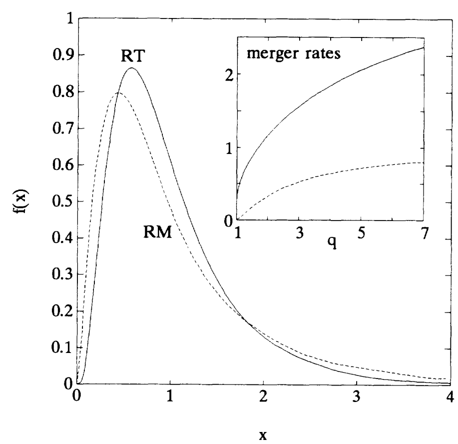

FIG. 2. Scale invariant bubble size distributions. The x axis is the bubble diameter in units of the average diameter. The full line is the result for RT fronts and the dashed line is for RM fronts. Inset: Dimensionless merger rates, $\omega_0(q)$, for the two instabilities. The upper line is for RT bubbles and the lower line is for RM bubbles.

the ensemble. This criterion has been previously suggested by Glimm and co-workers [7,8]. At coexistence, we expect, from dimensional arguments, that the bubble velocities are proportional to $\lambda^{1/2}$ in the RT case and to λ in the RM case. We thus define the effective wavelength ratio of the bubbles at their coexistence stage as $q=[u_1(t_1)/u_2(t_1)]^2$ for RT and $q=u_1(t_1)/u_2(t_1)$ for RM. The merger time is defined by $\tau(q)=t_2-t_1$, and the merger rate is $\omega(q)=\tau(q)^{-1}$. We denote by $\omega_0^{\mathrm{RT}}(q)$ the dimensionless RT merger rate. For two RT bubbles with wavelengths λ_1 and λ_2, and with a gravitational acceleration g, the merger rate is

$$\omega(\lambda_1,\lambda_2)=[g/(\lambda_1+\lambda_2)]^{1/2}\omega_0^{\mathrm{RT}}(\lambda_1/\lambda_2)\,.$$

This is due to the dimensional scaling of the flow equations: The equations are invariant under the transformation

$$\{\lambda\to w_1\lambda,\ \ g\to w_2 g,\ \ t\to(w_1/w_2)^{1/2}t\}$$

for any $w_1,w_2>0$. For RM bubbles, the merger rate scales as $\omega(\lambda_1,\lambda_2,t)=t^{-1}\omega_0^{\mathrm{RM}}(\lambda_1/\lambda_2)$, due to the invariance of the flow equations under the transformation $\{\lambda\to w\lambda,\ t\to wt\}$ in this case ($g=0$). Note that $\omega_0(\lambda_1/\lambda_2)=\omega_0(\lambda_2/\lambda_1)$. The dimensionless merger rates found are shown in the inset of Fig. 2 for both instabilities. For two identical bubbles ($q=1$) the merger rate is zero, since there are no mergers in a periodic array of equal bubbles. The merger rate increases for large q, reflecting the fact that very large bubbles quickly overtake their small neighbors. These merger rates belong to the class of merger rates, described in Ref. [11], for which the merger model reaches a scale invariant regime. This is verified by numerically solving Eq. (1). The size distribution function reaches the scale invariant form

$$g(\lambda,t)=N(t)\langle\lambda(t)\rangle^{-1}f(\lambda/\langle\lambda(t)\rangle)\,,$$

where $\langle\lambda(t)\rangle$ is the average wavelength. The scaled distribution function f is independent of the initial distribution. The asymptotic scaled size distributions found are shown in Fig. 2. The distributions are quite narrow, with few bubbles much larger than the average.

For the RT case, scale invariance implies a constant acceleration of the bubble interface height [11], $h(t)=\alpha g t^2$. We find $\alpha=0.051$. This value is in agreement with full numerical simulations of the RT front with random initial perturbations (Youngs [4] finds $\alpha=0.04$–0.05; Ofer and co-workers [6] and Glimm *et al.* [4] and $\alpha=0.05$–0.06) and experimental results (Read [3] obtained $\alpha=0.058$–0.065 in approximately two dimensional experiments).

The results for RM bubble fronts exhibit a new scale invariant growth behavior. Using Eq. (1), we find that the average wavelength increases as $d\langle\lambda\rangle/dt=t^{-1}\theta\langle\lambda\rangle$, where $\theta=\int_0^\infty\int_0^\infty\omega_0^{\mathrm{RM}}(x/y)f(x)f(y)dx\,dy$ is the average over the scale invariant distribution of the dimensionless merger rate. Using this to evaluate the average front velocity, $\langle u\rangle=[1/(3\pi)]\langle\lambda\rangle/t$, we find that the front height asymptotically increases as

$$h(t)\sim at^{\theta},\quad a=\lambda_0/3\pi\theta t_0^{\theta}\,, \tag{4}$$

where t_0 is an arbitrary time in the scale invariant regime, and λ_0 is the average wavelength at that time. The factor a in Eq. (4) depends on the initial perturbation, while the power law exponent θ does not. Using the merger law found above (Fig. 2) we find $\theta=0.40$. This prediction for power law growth of the RM bubble front has been checked by full hydrodynamic simulations of a single shock wave that passes through a randomly perturbed interface between two nearly incompressible fluids with a high density ratio [19]. It is found that the bubble interface rises as t^{θ} with $\theta=0.35\pm0.01$, compared with $\theta=0.40$ from the model. Further analysis of single-mode and multimode RM fronts using the model presented in this paper as well as full hydrodynamic simulations will be detailed in a forthcoming publication [19].

In conclusion, the model described above offers a description, based on the flow equations, of instability generated bubble front evolution. The RT front acceleration derived from the model is in good agreement with simulations and experimental results. We obtain the first description of the Richtmyer-Meshkov front in terms of bubble competition, and novel results for the RM bubble front evolution. In both cases, the front dynamics attain a scale invariant regime, with a fixed scaled bubble size distribution. The model may also be applied to calculate other aspects of the front evolution, such as the time to arrive at the scale invariant regime for different initial bubble distributions.

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[18] The model result is slightly higher, since the small bubble is washed into the spike region in the very late stages of competition, where the model equations cannot describe it correctly.

[19] U. Alon, J. Hecht, D. Ofer, and D. Shvarts (to be published).