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Chern-Simons Diffusion Layers? Short Wavelength Dynamics Determined by Gauge Invariance

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Uncoupled, classical Maxwell-Chern-Simons theory is studied in regions in which the Chern-Simons coefficient is a spacetime dependent quantity. Gauge invariance demands a modification of the strict Chern-Simons form in such a region. Behavior is governed by boundary and initial conditions. In many cases, the resulting *classical* behavior qualitatively resembles diffusive processes in statistical systems.

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Introduction.—The Abelian Chern-Simons action defined by [1–3]

$$S = \int dV \; \tfrac{1}{2}\mu \, \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \qquad (1)$$

is only gauge invariant if the coefficient μ is globally constant. In certain cases of interest the value of the Chern-Simons coefficient is determined by underlying properties of a physical system. For example, radiative corrections to Dirac fermions lead to a decisive value for the coefficient [4–6]. A natural question is what happens in such a theory if those physical properties change over some length scale important to the physics of the model. Then one is naturally left with a Chern-Simons coefficient which is a function of position and perhaps even time. However, since the action given in Eq. (1) is not gauge invariant under such circumstances it would appear that Eq. (1) cannot be a complete description of physics in such a region. The question to be addressed in this paper is the following: can the theory be made self-consistently complete in a region in which $\mu = \mu(x)$?

We may adopt one of two points of view: (i) Chern-Simons theory is only an effective theory which must always be derived from some deeper underlying model. The inadequacies of the Chern-Simons term for varying $\mu(x)$ must be resolved by a more careful investigation of the underlying theory. (ii) The Chern-Simons action is a valid starting point for constructing a theory. Its relation to physics is not *a priori* clear, but the requirements of gauge invariance and energy conservation will dictate how such a model should behave.

In support of the former viewpoint, one recognizes that the dynamically induced Chern-Simons term which arises from Dirac fermions of a definite spin is only part of the story. There are short wavelength corrections to the form in Eq. (1) which are usually ignored [6]. These will almost certainly have an important influence in any region in which physical quantities associated with fermions change over some relevant length scale.

For the latter viewpoint, curiosity inevitably prompts the question: what self-consistent solutions can be obtained from the Chern-Simons action as it stands? Do such solutions represent physics? Or perhaps even more interesting: are the requirements of gauge invariance and energy conservation enough to uniquely determine some universal kind of behavior? If, for example, we computed the appropriate effective action for Dirac fermions with an x-dependent chemical potential, would the answer be comparable to that obtained if we had simply started from a position dependent Chern-Simons coefficient and fixed the gauge invariance issue?

It will not be possible to answer this last question in the present Letter. Instead, the aim will be to use gauge invariance to determine self-consistent behavior. The outstanding issues will be discussed elsewhere.

Sharp junction.—Consider first the problem of a sharp steplike boundary at $x_1 = 0$ separating two regions of constant μ. The variation of the action with respect to

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the field A_μ yields the field equations and appropriate boundary conditions for the system

$$\partial^\mu F_{\mu\nu} + \mu\epsilon_{\nu\rho\lambda}\partial^\rho A^\lambda = 0, \tag{2}$$

$$\Delta[-F_{1\lambda} + \tfrac{1}{2}\mu\epsilon_{\mu 1\lambda}A^\mu] = 0. \tag{3}$$

Δ represents the change of a given quantity across the boundary at $x_1 = 0$. In component form,

$$\Delta B - \tfrac{1}{2}\Delta\mu A^0 = 0, \tag{4}$$

$$\Delta E_1 + \tfrac{1}{2}\Delta\mu A^2 = 0. \tag{5}$$

Note that the field has been assumed continuous through the boundary. As remarked in [7] this condition would be too restrictive in general since one might conceivably have a contact potential between the regions.

Although the boundary conditions are not gauge invariant, it is possible to extract gauge invariant results. A simple calculation shows that if one chooses $E_2 = 0$ on the boundary then all is well. More generally $(\partial^\nu\mu)F^*_\nu = 0$ assures gauge invariance. This condition is not very interesting in the static case since it simply implies that one must have perfectly reflecting boundary conditions; i.e., there is no contact between the regions. To obtain more interesting gauge invariant solutions more variables are needed [7].

If one examines the change in the Poynting vector parallel to the surface $\Delta S_1 = E_2\Delta B$, for example, it is seen that it is nonzero. In other words, energy is not conserved in passing through the barrier. ΔS_2 is not conserved either, but the contributions do not cancel. Since ΔB and $\Delta\mu$ are related for any A_0 through the boundary condition, it would seem that the difficulty is really that energy is not conserved. In a classical, descriptive theory this situation can be dealt with in a gauge invariant manner [7] either by the introduction of sources or time dependence.

It is noteworthy that the nonconservation of energy expresses something which might otherwise have been qualitatively obvious, namely, that such a sharp boundary would most likely not represent a situation of thermodynamical equilibrium. At the simplest level, particles on either side of the boundary would have different masses [1–3]. The pressure on either side of $x_1 = 0$ would therefore be different [7].

Finite thickness.—To avoid some of the conceptual difficulties of a sharp boundary and also to generalize the discussion somewhat, we shall now consider the following action:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\mu(x)\varepsilon_{\nu\rho\lambda}A^\nu\partial^\rho A^\lambda + f(x)J^\mu_S A_\mu. \tag{6}$$

This action should be general enough to obtain gauge invariant solutions. J^μ_S is an external source which can be used to balance the energy or gauge invariance equations. The function $f(\mathbf{x}, t)$ acts like a membrane of variable thickness which mediates the contact between the Chern-Simons system and the source. Under the transformation $A_\mu \to A_\mu + \partial_\mu\theta$, the action $S = \int dV\mathcal{L}$ changes like

$$\frac{\delta S}{\delta\theta} = -\tfrac{1}{2}[\partial^\nu\mu(x)]\varepsilon_{\nu\rho\lambda}\partial^\rho A^\lambda - [\partial_\nu f(x)]J^\mu_S, \tag{7}$$

where I have assumed that $\mu(x)$ and $f(x)$ are gauge invariant. For simplicity the spatial variation of $\mu(\mathbf{x}, t)$ and $f(\mathbf{x}, t)$ is purely in the x_1 direction. Invariance of the action then requires that

$$-\frac{1}{2}B\frac{\partial\mu}{\partial t} - \frac{1}{2}E^2\frac{\partial\mu}{\partial x_1} = -J_0\frac{\partial f}{\partial t} + J_1\frac{\partial f}{\partial x_1}. \tag{8}$$

We should be duly wary at this juncture. Equation (8) relates the Chern-Simons coefficient to the dual of the field tensor. Since the derivatives act on μ it is natural to solve this equation for μ as a function of the field components. But in doing this we are introducing "higher derivatives" into the theory. The resulting action is not really of the Chern-Simons form any longer. This is a clear indication that the original action was only valid for constant μ. Gauge invariance is telling us that higher derivatives are needed. Of course, we could choose to retain μ as the given quantity and eliminate E_2 and B, though this might be considerably harder to implement. One assumes that the same answer must result in either case.

In order for $f(x_1, t)$ to fulfill its purpose satisfactorily, it must satisfy certain boundary conditions. First, as $\partial_\mu\mu(x_1, t) \to 0$, we must have $f(x_1, t) \to 0$ to reinstate normal conservation laws for constant μ. Also $f(x_1, t) = 0$ when $E^2 = B = 0$, so that the extra terms become unnecessary in the special cases which are already gauge invariant. Consider two simple instances of Eq. (8).

(i) If $\mu(x, t)$ is a time-dependent variable the external source term is not formally required in the x-dependent region. In that case Eq. (8) is separable and in the varying region has a solution of the form

$$\mu(x_1, t) = \overline{\mu} + \mu_0 e^{-\frac{\gamma}{B}t} e^{\frac{\gamma}{E^2}(x_1 - x_0)} \tag{9}$$

for constant E^2, B. [This equation has infinitely many nonseparable solutions of the form $\mu(E_2^{-1}x_1 - B^{-1}t)$ when E^2 and B are constants, but for nonconstant fields separability is an issue. See below.] Here γ is a real separation constant and $\overline{\mu}$, μ_0, x_0 are to be fixed from the boundary conditions. If the fields E^2 and B are not constant in space and time, but are themselves separable functions then setting $\mu(x_1, t) = h(t)g(x_1)$ gives that $h(t)$ and $g(x_1)$ satisfy Langevin's equation. The equation for $h(t)$, for example, takes the generic form

$$a\frac{dh(t)}{dt} + bh(t) = F(t), \tag{10}$$

where $F(t)$ is defined through the Taylor expansion of the electric and magnetic fields. This clearly has the solution

$$h(t) = a\int_0^t d\tau e^{\frac{b}{a}(\tau - t)}F(\tau) + h_0 e^{-\frac{b}{a}t}. \tag{11}$$

The behavior will essentially be diffusive, exponential decay type behavior, modulated by the form of $F(t)$.

In either case the gradient of the Chern-Simons term diffuses away until it reaches some constant value. This is rather typical of the coarse-grained, diffusive behavior of thermodynamic quantities outside of equilibrium, which is suggestive in view of the energy arguments of the previous section. In a system which possesses asymptotic regions of constant μ, the form of $\mu(x)$ must clearly yield to the relevant boundary conditions in those regions. This will almost certainly preclude constant E_2 and B and require the diffusive behavior of Eq. (11). On the other hand, in an infinite, one-dimensional system with constant E_2 and B longitudinal waves or any other kind of normalizable behavior is allowed.

(ii) If we wish to sustain a time-independent gradient in μ then we must include a nonzero coupling to the source J_S. Note that, in general, $f(\mathbf{x},t)$ will be time dependent even though μ is not, but for constant E^2 a solution which satisfies the above boundary conditions may be written $f(x_1) = \frac{\alpha}{2}\frac{d\mu}{dx_1}$, for some constant α. Substituting into Eq. (8) we obtain

$$\nabla^2\mu(x_1) = D\frac{d\mu(x_1)}{dx_1}, \tag{12}$$

$$\frac{df}{dx_1} + Df(x_1) = 0, \tag{13}$$

where $D = E^2/\alpha J_1$. Equation (12) resembles a diffusion equation and Eq. (13) is manifestly the exponential decay law.

In order to obtain field equations satisfying this restriction we must use this equation in two ways. First, it is a restriction on the values of the related quantities. Since the derivatives act on μ it is natural to regard this as a restrition on μ, though there is a certain amount of ambiguity here. A self-consistent solution is required. The second thing we must do with Eq. (8) is to impose it as a condition on the variation of the fields A_μ. Since the equation is an A_μ-dependent function, the variation of this condition must vanish.

Field equations.—The next step is to determine the field equations. In general this will be a rather difficult problem, since μ will be a nonlocal function of the field strengths. We shall consider only an example. In the special case in which E_2 and B are both constant throughout the (x_1,t) varying region, solutions are straightforward to obtain. The Bianchi identity $\partial_2 E_1 = 0$ coupled with the constant field condition implies that all quantitities are now x_2 independent. The gauge invariance condition is satisfied by an arbitrary function of the combination $\gamma(E_2^{-1}x_1 - B^{-1}t)$. The only nontrivial, nonsingular solution to the variation of the gauge invariance equation is given by $\delta A_2 = 0$. Using this, one finds that $\delta\mu = 0$. The field equations for the remaining components of A_μ are then

$$\partial_\mu F^{\mu 0} + \mu(x)B + \tfrac{1}{2}(\partial_1\mu)A_2 = 0, \tag{14}$$

$$\partial_\mu F^{\mu 1} + \mu(x)E_2 - \tfrac{1}{2}(\partial_0\mu)A_2 = 0. \tag{15}$$

Combining these leads to

$$E_2(\partial_1 E_1) + B(\partial_t E_1) = 0, \tag{16}$$

which clearly has the general solution

$$E_1 = E_1[\gamma(E_2^{-1}x_1 - B^{-1}t)]. \tag{17}$$

From Eq. (5) it seems natural to choose $E_1 \propto \mu$. As noted above, the situation of constant E_2 and B is a very special case. If we have as a physical picture a region of varying μ in between regions of spatially constant μ, then Eq. (4) suggests that the magnetic field should also have followed the contour of μ. Because of Eq. (11), such a solution would again have to be implemented self-consistently. It is only speculated here that, owing to the form of Eq. (11), decaying gradients in all the fields will be the natural outcome in general. The generalization for nonconstant E_2, B will be pursued elsewhere.

In connecting E_2, B, and μ we have eliminated the dependence on μ as a free parameter. One sees that its spacetime variation must be in accord with the changing value of the field. Only the constant multipliers μ_0 and γ survive as free scales in the theory. To proceed one must specify the variation of E_2, B, or μ in a self-consistent way. From Eq. (4), one might perhaps expect "natural" behavior to result from $B \propto \mu$ in a region varying between flat regions (a junction). The problem becomes one of specifying boundary conditions.

It would be interesting to push this idea of self-consistent behavior based on gauge invariance (and boundary conditions) to see just how far it leads. Of course, this might be a dead end and it might lead to some kind of universal behavior. A more geometric analysis of Eq. (8) could help here. The diffusive type of behavior found here might only be a special case but it is an appealing physical notion which seems to elevate μ from the role of being purely a mass or statistics parameter to being a thermodynamical quantity, that is, a continuous classical field which flows so as to reach some final constant value. This would be of importance in a wide range of physical systems involving parity breaking. The solutions described in this short Letter are of a general nature but it is possible that the idea could be put on a firmer basis in the context of some particular system. Since the Chern-Simons coeffient for fermions at finite density is a function of the chemical potential, it would not be unnatural to construct a model like a p-n junction. This work is in progress. A different though nevertheless interesting situation in a quantized theory was discussed in Ref. [8] where a gauge variant parity breaking term was discussed. There the resolution to the lack of invariance was to compute nonperturbatively. Clearly this is not the resolution to the problems in this work, but a further investigation might lead to some clues which could help here. It remains to be seen how the present model would respond to being quantized. It is emphasized, however, that the behavior descibed in this Letter is purely at the classical or "effective" level. The message of the present Letter is that a position-dependent Chern-Simons term naturally leads to nonequilibrium behavior even at the classical level.

Finally, it is interesting to remark that the Chern-Simons coefficient is normally quantized for non-Abelian gauge fields [1,3]. This requirement is rescinded if μ is a varying field. However, in asymptotic regions for which μ is constant it would have to be quantized according to the usual conditions. This leaves open several interesting lines of investigation.

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