Failure Probability and Average Strength of Disordered Systems

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Using a new recurrence-relation method, we have calculated the failure probability and average strength of random systems of up to linear dimension $L=5000$. We find a *deep minimum in the failure probability at an optimal sample size* (L_0). As the applied stress decreases the depth of this minimum *grows exponentially* and L_0 increases *algebraically*. At large sample sizes the *average strength* exhibits a logarithmic size effect, in contrast to recent suggestions of algebraic scaling in related models.

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The tensile strength of materials is usually orders of magnitude weaker than that expected from the strength of the atomic bond. The reason for this reduction in strength is that defects, such as cracks and dislocations, nucleate and cause fracture at applied stress levels far below that of a defect-free system. A related effect is that the average strength of materials often shows a size effect, in which larger samples have lower fracture stress than smaller samples. Thus extremely strong fibers and whiskers exist, but it is very difficult to make larger samples with similar strengths. The size effect in material strength is closely related to the fact that tensile fracture, and failure in other contexts, is often a heterogeneous nucleation problem, in which the failure instability initiates in an especially weak or highly stressed part of the material. The observed properties are thus dominated by *rare* or *extreme* fluctuations in the geometry and stress field in the materials. A similar dependence on rare fluctuations has recently been suggested in the depinning of charge-density waves.

New insight into the interplay between disorder and failure has followed the development of numerical algorithms to simulate these processes [1], and scaling theories which explicitly take into account the dependence of failure on rare fluctuations [2] (for the application of similar ideas to charge-density-wave depinning see [3]). Because of the anomalous nature of the fluctuations, it is important to study the appropriate *distribution functions,* and here we concentrate on one key distribution, the *failure probability.* However, the algorithms available for spring and electrical networks are quite slow computationally, and are not able to convincingly test the asymptotic *finite-size-scaling* forms for the probability of failure or the average failure strength. We have thus investigated a simpler *chain-of-bundles* model (see below) for the failure of heterogeneous networks, and have compared the results of these calculations with new simulations of $L\times L$ square-lattice fuse networks of up to size $L=100$. Despite its simplifications the chain-of-bundles model is still extremely complex, and thus far a sophisticated transition-matrix method has been most useful in analyzing its behavior [4]. However, this transition-matrix method does not allow a complete study of the key finite-size-scaling behavior. We have thus developed a *powerful new recurrence relation for the failure probability* and using this recurrence relation we are able to calculate the failure probability and average strength for lattices of up to $L=5000$.

Consider a square lattice in which each bond of the lattice is assigned a strength (e.g., tensile strength, critical current, etc.), where for concreteness we discuss in detail the case of a uniform distribution of bond critical currents extending from the origin to a width W. Prior to failure, all bonds have the same transport modulus (e.g., spring constant, conductance) where for concreteness, we take the case of an electrical system of conductance g. When a bond has failed, its transport modulus is set (irreversibly) to zero. An external current is injected into the lattice, and if the current is sufficiently high, the whole network will fail. At sufficiently low applied current, some of the bonds in the network fail, but the network eventually stabilizes with a flaw population generated by the failure process. The lowest applied current for which the system undergoes complete failure is the critical current (similarly for the critical stress). Doing many configurations leads to a probability distribution, $F_n(i)$, which is the probability that at applied current i a sample of size n $(=L^2)$ will fail. As in our earlier simulations of the diluted fuse network [2], we use the conjugate-gradient technique with an accuracy of 10^{-12} for the squared residual.

In some high-strength fiber-reinforced composites [4] the failure behavior is well described by quasi-one-dimensional models which consider only linear cracks. In fact, these *chain-of-bundles models* have been very successful in describing both the current distribution [5] and failure properties [2,4,6] of higher dimensional systems. For example, to model a two-dimensional $L_x\times L_y$ square lattice, we divide the square lattice into a *chain* of L_y 1D models like that in Fig. 1. The restriction to linear cracks

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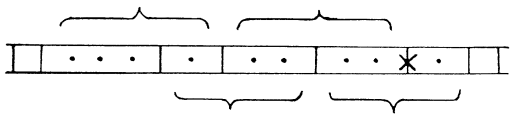

FIG. 1. The one-dimensional model after some bonds have broken. A bracket indicates the failed bonds which contribute to the current or stress enhancement at the surviving bond inside the bracket.

is equivalent to assuming that the current injected into each of these 1D models is the same, and that each bundle (1D model) is independent of all the others. If we define the failure probability of one bundle to be $f_{L_x}(i)$ [its survival probability is $s_{L_x}(i)=1-f_{L_x}(i)$], then the failure probability of L_y such bundles in series (the chain-of-bundles model) is simply $F_n(i)=1-[1-f_{L_x}(i)]^{L_y}$. The difficult part of the analysis is to calculate $f_L(i)$, and this is the quantity we calculate using our new recurrence relation method.

A key input in these 1D models is the amount of current or stress enhancement which occurs in surviving bonds which have neighboring failed bonds (e.g., the bond marked with an × in Fig. 1). The local-load-sharing rule which we use is $i_k=i(1+k/2)$, where i is the applied current per bond and i_k is the current in a bond surrounded by k failed bonds (e.g., for the bond marked × in Fig. 1, $k=3$). This law has been extensively used in modeling composites (with i replaced by applied tensile stress [4]), and is consistent with results for 2D resistor networks where a similar current enhancement [up to $\ln(k)$ corrections] occurs when two cracks are close to each other [2].

To analyze the one-dimensional model of Fig. 1, let a {1} denote a surviving bond and a {0} denote a failed bond. Then for $L=2$, the survival configurations are {11,10,01}. For arbitrary L, there are 2^L-1 survival configurations and one failure configuration {00000...000}. The probability that a bond survives when it has k failed neighbors, W_k, is given by $W_k=1-\int_0^{(1+k/2)i}p(i')di'$, where $p(i)$ is the differential failure probability of a bond [e.g., for a uniform distribution we use $p(i)=1/W$ for $0\le i\le W$]. It is then straightforward to write down the survival probability for each survival configuration on a lattice of size L in terms of W_l and f_l [$=f_l(i)$], with $l\le L$ (e.g., for $L=2$, the survival probabilities are W_0^2, W_1f_1, W_1f_1). Although we can then calculate the failure probability of systems of size L using data on smaller systems (e.g., $f_2=1-W_0^2-2W_1f_1$), this algorithm is slow as it scales as the number of survival configurations (i.e., as 2^L). We thus developed a more efficient recurrence relation.

To develop this method, we separate the full set of 2^L-1 survival configurations into judiciously chosen subsets. Let a *lone fiber* be a surviving fiber which is surrounded by failed fibers (e.g., the fiber marked with an × in Fig. 1). Then, let $\{A\}$ be the set of all survival configurations which contain only failed fibers *and* lone fibers, *and* which are bracketed at both ends by lone fibers. The small L members of this set are {101,1001, 10001,10101,...}. $\{B\}$ is the same set as $\{A\}$, with the exception that one (specified) end of the configuration must be failed. The small L members of this set (for the case where the left end is failed) are {01,001,0001, 0101,00101,...}. There is a complementary set to $\{B\}$ which has the same probability but which has the failed ending on the right. The generating functions for $\{B\}$ (see below) and its complement are the same, so we do not distinguish between them. $\{C\}$ is the same as $\{A\}$ except that *both* ends have failed. The small L members of this set are {010;0100,0010;01010,...}. Finally, we define $\{p\}$ to be the set of configurations which contain no failed bonds $\{(\cdot),1,11,111,1111,...\}$, where $(\cdot)$ is the empty set. Associated with these classes of survival configurations are the generating functions,

$$A(z)=\sum_{L=3}^{\infty}A_Lz^L,\quad B(z)=\sum_{L=2}^{\infty}B_Lz^L,\quad C(z)=\sum_{L=3}^{\infty}C_Lz^L,$$

where A_L, B_L, and C_L are the sums, respectively, of the survival probabilities of the sets $\{A\}$, $\{B\}$, and $\{C\}$ of fixed sample size L. The generating function for the set $\{p\}$ is

$$p(z)=\sum_{L=0}^{\infty}(W_0)^Lz^L=\frac{1}{1-W_0z}. \tag{1}$$

Some study [7] shows that the generating function for the survival configurations $s(z)=\sum_L s_Lz^L$ is related to $A(z)$, $B(z)$, $C(z)$, and $p(z)$, so that

$$s(z)=C(z)+\frac{[1+B(z)]p(z)[1+B(z)]}{1-p(z)A(z)}. \tag{2}$$

It is also true that

$$f_L=1-s_L$$

implies

$$f(z)=\frac{1}{1-z}-s(z), \tag{3}$$

where we have used $f_0=0$, $s_0=1$, and $f(z)=\sum_Lf_Lz^L$. Combining Eqs. (1)–(3), we get the key generating function equation,

$$(1-z)[1+B(z)]^2-[1-W_0z-A(z)]\{1-(1-z)[f(z)+C(z)]\}=0. \tag{4}$$

We then expand this identity in powers of z and by setting the coefficient of the z^L term to zero we find the recurrence relation

$$X_L=X_{L-1}+W_0\nabla_{L-1}X-2\nabla_LB-A_L+f_1A_{L-1}-B_2B_{L-2}+\sum_{m=1}^{L-4}(A_{m+2}\nabla_{L-m-2}X-B_{m+1}\nabla_{L-m-1}B), \tag{5}$$

where $X_L = f_L + C_L$, and $\nabla_L Y = Y_L - Y_{L-1}$. To implement this recurrence relation, we need expressions for the survival probabilities A_L, B_L, and C_L. We have also found recurrence relations for these probabilities. This we have done by defining the subset, $\{c_{L,l}\}$ of $\{C\}$, which is the set of survival configurations of length L which *end* with exactly l failed bonds. For example, $\{c_{7,2}\} = \{0010100, 0100100\}$. Similarly, we define the subset $\{b_{L,l}\}$ of $\{B\}$ and the subset $\{a_{L,l}\}$ of $\{A\}$. The survival probabilities of these subsets are then related to the survival probabilities of subsets of shorter length via the recurrence relations

$$b_{L,l} = f_l W_l \delta_{L-l-1} + \sum_{r=1}^{L-l-2} b_{L-l-1,r} W_{l+r} f_l , \tag{6}$$

$$c_{L,l} = f_l W_{L-l} f_{L-l-1} + \sum_{r=1}^{L-l-3} c_{L-l-1,r} f_l W_{l+r} , \tag{7}$$

and

$$a_{L,l} = b_{L-1,l} W_l . \tag{8}$$

Finally, we find A_L, B_L, and C_L by summing over l (for example, $A_L = \sum_l a_{L,l}$). Equations (5)–(8) are the exact recurrence relations for the failure probability $[f_L(i)]$ of the 1D model, and the time required to evaluate them scales at most as L^3. In fact, in the case of distributions which have an upper cutoff (such as the uniform distribution), it is possible to reduce the computations to a number which scales as L (at large L). Brute-force calculations are restricted to systems of order $L = 20$, while with the recurrence relations above, we have studied systems of up to $L = 5000$ on a Sun workstation.

Results for the size dependence of the failure probability for fuse networks and for the chain-of-bundles model are presented in Fig. 2. For high applied current, the failure probability increases monotonically with system size, but for low values of applied current it shows a *deep minimum at an optimal sample size*, which we define to be (L_0). It is not possible to probe this minimum fully for the fuse network, as this would require many thousands of configurations on sample sizes much greater than $L = 20$. Nevertheless, the data in Fig. 2(b) for $i/W = 0.05$ clearly show the rapid *decrease* in failure probability characteristic of the onset of the deep minimum in Fig. 2(a). For the uniform distribution it is possible to estimate the location and depth of the probability minimum as follows. Since the distribution has width W, and the current enhancement in a bond obeys $i_k = i(1 + k/2)$, there can certainly be no cluster of failed bonds of size greater than L_{0a}, given by $W = i(1 + L_{0a}/2)$, which implies $L_{0a} = 2W/i - 2$. Once a sample exceeds this size, the number of survival configurations no longer increases at the same rate, and so (roughly) we would expect the failure probability to increase. The data for the chain-of-bundles model is broadly in agreement with this behavior. In the regime prior to the minimum in Fig. 2, the effect of local stress concentrations is relatively weak,

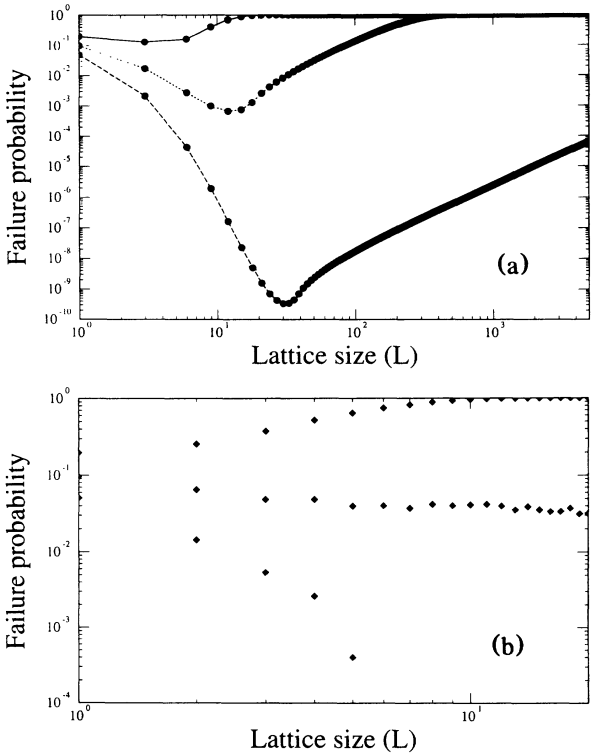

FIG. 2. The size dependence of the failure probability. (a) Data for the chain-of-bundles model. We used $F_n(i) = 1 - (1 - f_L)^L$ with $n = L^2$, where f_L is found using the recurrence-relation method. (b) Data for the fuse network, for $L \times L$ square lattices using 5000 realizations at each value of $L = 1$–20. In both figures, starting from the top the data are for $i/W = 0.2$, $i/W = 0.1$, and $i/W = 0.05$.

so the failure probability can be estimated by assuming uncorrelated bond breaking, which gives $f_L(i) \sim [f_1(i)]^L = (i/W)^L$. For small i/W, the depth of the probability minimum $F_{\min}$ can thus be estimated by replacing L in this equation by L_0 which gives $F_{\min} \sim 1 - [1 - (i/W)^{L_0}]^{L_0}$. Using $L_0 \sim W/i$ and dropping constant factors yields $F_{\min} \sim (i/W)^{W/i}$. Although this argument overestimates the depth of the probability minimum, it indicates that as i/W approaches zero, $F_{\min}$ approaches zero *exponentially*, as observed in the numerical data. This probability minimum can be expected in other random systems (e.g., we have checked that it occurs in systems with a Weibull distribution of local failure thresholds [7]) and is an important feature of these problems which has been previously inaccessible to direct calculation. It also raises the intriguing possibility of designing materials and structures so that they are used near this minimum, with a consequent minimization of their probability of failure.

The average strength of fuse networks and of the chain-of-bundles model are presented in Fig. 3. It is seen that the data appear inconsistent with an algebraic size

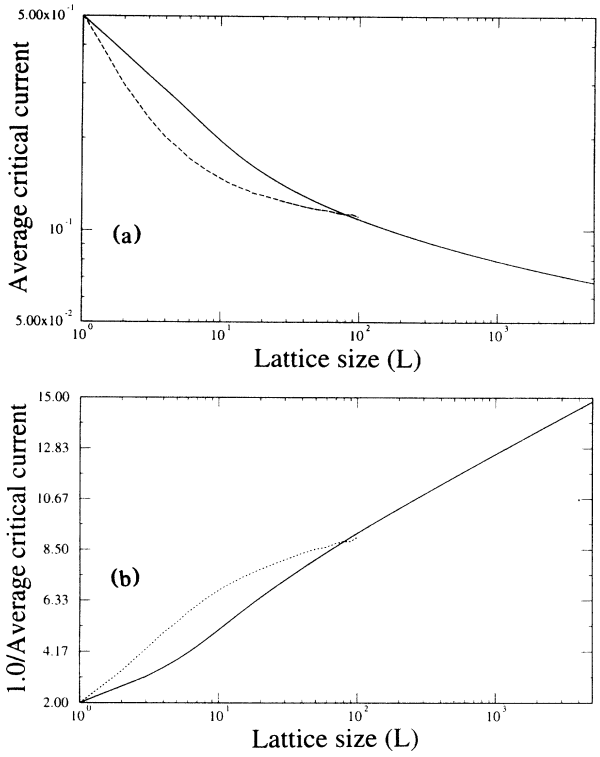

FIG. 3. The average critical current of heterogeneous systems with a uniform distribution of local-failure thresholds. Data for the chain-of-bundles model (——) and for the random-fuse network (---). Data for the chain-of-bundles model are exact to the resolution of the figure. The fuse-network data are found from an average over 5000 configurations for $L=1-20$, and an average over 25 configurations for $L=30$, 40, 50, 60, 70, 80, 90, and 100. (a) Test of algebraic scaling and (b) test of logarithmic scaling.

effect both for the chain-of-bundles model and for the fuse network. If we do fit the fuse network data from $L=10$ to $L=100$ to an algebraic law, we get an exponent of about -0.1 as found in previous work using data on a restricted set of sample sizes [8]. The data for the chain-of-bundles model (solid line in Fig. 3) are quite close to the form $(1+k\ln L)^{-1}$ over a broad range of system sizes. The fuse network data (dotted line in Fig. 3) can be fitted to a size effect of the form $[1+k(\ln L)^{a}]^{-1}$, with $\frac{1}{2}<a<1$ as suggested by scaling theories [10]. However, the numerical data for the fuse network are not sufficiently accurate and extensive to convincingly distinguish between an algebraic law (with a small exponent) and a logarithmic law at large sample sizes. Nevertheless, the fact that scaling arguments [9,10] and the results for the chain-of-bundles model support a logarithmic law make it likely that logarithmic scaling is correct for the fuse network. Since these arguments also apply, with only notation changes, to the tensile failure of beam and spring models [11], we suggest that those models also exhibit a behavior like that found here.

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[1] L. de Arcangelis, S. Redner, and H. J. Herrmann, J. Phys. (Paris), Lett. **46**, L585 (1985).
[2] P. M. Duxbury, P. D. Beale, and P. L. Leath, Phys. Rev. Lett. **57**, 1052 (1986); P. M. Duxbury, P. L. Leath, and P. D. Beale, Phys. Rev. B **36**, 367 (1987).
[3] S. N. Coppersmith, Phys. Rev. Lett. **65**, 1044 (1990).
[4] D. G. Harlow and S. L. Phoenix, Int. J. Fracture **17**, 601 (1981).
[5] B. Kahng, G. G. Batrouni, and S. Redner, J. Phys. A **20**, L827 (1987).
[6] W. A. Curtin and H. Scher, Phys. Rev. Lett. **67**, 2457 (1991).
[7] P. L. Leath and P. M. Duxbury, Phys. Rev. B (to be published).
[8] L. de Arcangelis and H. J. Herrmann, Phys. Rev. B **39**, 2678 (1989).
[9] R. L. Smith, Proc. R. Soc. London A **382**, 179 (1982).
[10] B. Kahng, G. G. Batrouni, S. Redner, L. de Arcangelis, and H. J. Herrmann, Phys. Rev. B **37**, 7625 (1988); P. M. Duxbury and S. G. Kim, Mater. Res. Soc. Symp. Proc. **207**, 179 (1991).
[11] H. J. Herrmann, A. Hansen, and S. Roux, Phys. Rev. B **39**, 637 (1989); A. Hansen, S. Roux, and H. J. Herrmann, J. Phys. (Paris) **50**, 733 (1989).