Weak-Noise Limit of Stochastic Resonance

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The autocorrelation function of a periodically driven bistable system is considered in the limit of weak noise. We predict a broad window in the Fourier spectrum with a universal (power law) decay of the spectral density, accompanied by dips of finite width at the second and fourth harmonics of the driving frequency. For higher driving frequencies the dips are replaced by peaks of finite though extremely narrow width. Analytical results are confirmed by numerical studies and are used to analyze previous experimental data.

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The investigation of periodic perturbations in noisy nonlinear systems is presently in the limelight of both experimental and theoretical studies. Of special interest is the so-called *stochastic resonance*—a phenomenon due to synchronization of hopping or threshold crossing with external modulation. Such effects have been discussed in a geophysical context [1], for a ring laser [2], for sensory neurons [3], and for a variety of other applications [4]. This stimulated both theory [5–8] and analog simulations [9–12].

Typically, the main concern of any theoretical description of stochastic resonance is the autocorrelation function or the spectral density. The difficulty is that even in the simplest case of a Gaussian white noise, when the process can be described by a one-dimensional Fokker-Planck equation, there exists no exact expression for those functions. Thus, the search for rigorous analytical approximations becomes pertinent. One can single out two situations when the analytical description can be performed more or less completely. The first is the case of linear response (LR) where the smallest parameter is the amplitude of the driving force, A [5]. The second is the weak-noise (WN) limit with the small parameter being the noise intensity, D. Related studies here are rather sparce [6,8], and, to our knowledge there has been no consistent analytical study of the autocorrelation function up to this date. The case when both the WN and the LR conditions are satisfied, i.e., $D\to 0$ and $A/D\to 0$, was examined in Refs. [5(a)] and [7]. The goal of the present study is the WN consideration far beyond the LR approximation, i.e., for $A/D\to\infty$. We wish to emphasize that the WN limit presents an opposite limit as compared to the LR approximation. Therefore, one can expect qualitatively different results for the former limit of weak noise.

Consider a bistable dynamical variable $x(t)$ subject to a white noise. Our concern is the standard autocorrelation function

$$S(t,\tau)\equiv\langle x(t+\tau)x(t)\rangle-\langle x(t+\tau)\rangle\langle x(t)\rangle . \tag{1}$$

Note that this definition differs from that of Ref. [5(a)] by the product of average values, but otherwise we will use the same notations. This definition of $S(t,\tau)$ removes all periodic components, so that no delta peaks are expected in the spectral density. Thus, we focus on the *decay* of the autocorrelation function with τ. For small τ this decay is due to fast equilibration near the current stable (or metastable) values of the random variable, $x_\pm(t)$. We will be mostly interested in larger time scales when the decay of $S(t,\tau)$ is due to hopping between the two stable values. In the low-noise limit the quasiequilibrium probability distributions are sharply peaked near the values $x_+(t)$ and $x_-(t)$. Thus, one can employ the two-state rate equation for the probabilities $n_\pm(t)$ to occupy the corresponding site [5(a)]:

$$\dot n_\pm=W_\mp(t)n_\mp-W_\pm(t)n_\pm . \tag{2}$$

The transition probabilities, $W_\pm(t)$, in the weak-noise limit have sharp maxima at the instant when the corresponding escape time is the smallest. Thus, in the leading asymptotic approximation they can be approximated by delta peaks:

$$W_\pm(t)=\alpha\delta(t-m\pm\tfrac{1}{4}),\quad |m|=0,1,\dots , \tag{3}$$

where time is measured in the units of the modulation period. The calculation of α in the above equation for a particular system should not cause any difficulties as it can be done within the adiabatic approximation [6,8]. We only note here that α increases with noise and decreases with the driving frequency.

From Eqs. (1)–(3) one can obtain the following expression for the autocorrelation function:

$$S(t,\tau)=x_+(t)x_+(t+\tau)n_+(t)n_-(t)G(t,\tau) , \tag{4}$$

where, for brevity of notations, we put $x_-\equiv 0$. In the above $n_\pm(t)$ are the equilibrium (large time) solutions of Eq. (2), while $G(t,\tau)$ is given by

$$G(t,\tau)=\exp\{\alpha([t+\tfrac{1}{4}]-[t+\tau+\tfrac{1}{4}]+[t-\tfrac{1}{4}]-[t+\tau-\tfrac{1}{4}])\} . \tag{5}$$

Here $[t]$ is the integer part of time reduced by the modulation period. The product of the equilibrium distributions in Eq. (4) can be derived from the fact that the

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values of $n_\pm(t)$ change only during the hopping instant. The values of the discontinuities during hopping, say from x_+ to x_-, in terms of $\mu = e^{-\alpha}$ are given by $\mp n_+(1-\mu)$ for n_+ and n_-, respectively. Together with the condition $n_+ + n_- = 1$ this leads to $n_+(t)n_-(t) = \mu/(\mu+1)^2$. Note that this result is time independent, unlike the values of n_+ and n_- taken separately. The product $x_+(t)x_+(t+\tau)$ in Eq. (4) changes periodically around some average value. This change, however, is not related to hopping but arises due to modulation of the equilibrium positions. Being less interested in such effects, we shall treat them later as corrections—they really turn out to be vanishingly small in a broad range of frequencies. For the time being we replace the aforementioned product by x_+^2 at the hopping instant. This means that the time dependence of the autocorrelation function is completely determined by the function $G(t,\tau)$. After averaging over t one obtains

$$\bar{S}(\tau) = \frac{\mu}{(\mu+1)^2} x_+^2 \bar{G}(\{\tau\}) \mu^{2[\tau]}, \tag{6}$$

where $\{\tau\} = \tau - [\tau]$ is the fractional part and

$$\bar{G}(y) = \begin{cases} 1 - 2y(1-\mu), & y \le \frac{1}{2}, \\ 2\mu - \mu^2 - 2y\mu(1-\mu), & y > \frac{1}{2}. \end{cases} \tag{7}$$

At $\{\tau\} = 0$ and $\{\tau\} = \frac{1}{2}$ the autocorrelation function has cusps. One can expect that being repeated periodically this *nonanalyticity* will lead to a rather peculiar structure of the spectrum. When performing the Fourier transformation of Eq. (6) one can integrate within a period with subsequent summation of a geometric progression. After some transformations one ends up with

$$\bar{S}(\omega) = x_+^2 \frac{2\mu(1-\mu)}{(1+\mu)\omega^2} \frac{\Omega}{2\pi} \times \frac{1-\cos(\pi\omega/\Omega)}{[1-\mu\cos(\pi\omega/\Omega)]^2 + \mu^2\sin^2(\pi\omega/\Omega)}, \tag{8}$$

where we explicitly restored the driving frequency Ω. This expression, together with the forthcoming estimations of the region of its validity and corrections, represents the main results of the present study. The crucial point is that on average the spectral density (8) is practically independent of the driving frequency. For μ close to 1 (i.e., $1-\mu \simeq \alpha$) the product $(1-\mu)\Omega$ equals [with $\frac{1}{2}(x_+ \cdots x_-) \equiv 1$] $(D/A)^{1/2}\omega_K^{\max}$ with $\omega_K^{\max}$ being the maximal Kramers frequency (this is discussed below in greater detail). In this sense the power-law decay, $\bar{S}(\omega) \sim \omega^{-2}$ is "universal," and it is remarkable that the modulation preserves this property of the unmodulated system, despite the fact that the absolute values of the spectral density may be increased by modulation by several orders of magnitude. The periodic factor in Eq. (8) which modulates the power-law decay has zero values at each even multiple of the driving frequency. These imply unlimitedly deep dips in the logarithmic scale. The zeros evidently arise due to the aforementioned nonanalyticity of the autocorrelation function, and, qualitatively, they also emerge from the "random hopping" approximation [12]. A less trivial question, however, is to what extent the dips survive for the not extremely low-noise conditions assumed in Eq. (8). Note that the dips in the spectral density are not related to any symmetry of the system. Hence, they are extremely vulnerable with respect to an increasing of the intensity of noise (or of other perturbations disregarded so far) which in turn destroys the nonanalyticity of the autocorrelation function. For that reason it is not surprising that only in a very limited number of experimental studies (see below) were such dips actually detected; moreover, even in those experiments the dips vanish when the noise is increased [13].

To assess the limitations of Eq. (8) we consider the example of a modulated symmetric quartic potential

$$U(x) = -x^2/2 + x^2/4 - Ax\sin(\Omega t). \tag{9}$$

Upon increasing of the noise intensity, D, the sharp steps in the autocorrelation function $S(t,\tau)$ are eroded to have a finite width $\delta t \sim (A/D)^{-1/2}\Omega^{-1}$. Associated with this time scale is a characteristic frequency $\omega_0 \sim (\delta t)^{-1}$. For $\omega \ll \omega_0$ the above analysis holds. Particularly, one can expect that those even multiples of the driving frequency which are smaller than ω_0 will lead to distinct though finite dips in the spectral density. In contrast, when performing the Fourier transformation of the autocorrelation function at higher frequencies, one will "analyze" the intervals δt with too fine resolution and will not notice its nonanalyticity. Thus, no peculiarities in the spectrum are expected here. The condition for the existence of the mth dip, $\omega_0 \gg 2m\Omega$, may be thus approximately expressed as

$$A/D \gg (2m)^2, \quad m = 1,2,\ldots. \tag{10}$$

Note that the ratio of A/D should be sufficiently large to observe at least one dip. This explains why such dips are absent in the LR approximation which considers the limit $A/D \to 0$. On the other hand, in realistic experimental and computational situations the values of A/D are limited by a few tens, as otherwise the Arrhenius factor becomes too small. From the condition (10) one thus concludes that while the dip at the second harmonic of the driving frequency is representative, the dip at the fourth harmonic will be on the verge of detectability and dips at higher harmonics can hardly be present in a realistic experimental situation.

To consider the *low-frequency limitation* of the above treatment recall that the asymptotic treatment implicitly assumes that all nonasymptotic parameters have moderate, finite values. Roughly speaking, that means that the frequency is expected to be of the order of the largest Kramers frequency, $\omega_K^{\max}$. More accurately, note that α in Eq. (3) can be estimated as $(D/A)^{1/2}\omega_K^{\max}/\Omega$. Large values of α mean that the particle is guaranteed to escape the well when the barrier is the smallest. This, however, also means that during the preliminary stage of modulation the chances to escape the well are also high. Thus, at low driving frequencies $\Omega \ll \omega_K^{\max}(D/A)^{1/2}$ the discrete hopping approximation will not work, in complete analogy with a strong increase of D. No dips are expected here.

To evaluate the restrictions at *large* Ω, first of all recall the neglect of the time dependence of $x_+(t)$ in Eq. (4). For small A these values are modulated around average values like $A^2\cos(2\Omega t)$. Taken separately, however, these corrections will not contribute to the time-averaged autocorrelation function. Only the higher term, namely, $A^4\cos(2\Omega t)\cos[2\Omega(t+\tau)]$ may become important. In the leading order in α this fourth-order term results in the following correction, $\bar{S}_1(\omega)$, to the spectral density:

$$\bar{S}_1(\omega) \sim \frac{A^4\alpha\omega\sin(2\pi\omega/\Omega)}{[\omega^2-(2\Omega)^2][\sin^2(\pi\omega/\Omega)+\alpha^2]}. \tag{11}$$

We do not present the numerical coefficient which, unlike Eq. (8), is specific solely for the symmetric quartic potential. Note that at small $\alpha\Omega$, there exists a sharp peak at $\omega=2\Omega$ with the height $\sim A^4/\alpha\Omega$ being independent of the driving frequency (recall that α is proportional to Ω^{-1}—see above). Comparing the height of the peaks with the main approximation given by Eq. (8), one concludes that for $\Omega \gg \omega_K^{\max}(D/A)^{1/2}A^{-2}$ the peaks will become detectable, and definitely will prevail over the dips at the same frequency. For higher approximations in A similar peaks are expected at higher even multiples of the driving frequency. An important point is that the peaks are intrinsically *finite*. Thus, their existence does not contradict the general conclusion of the absence of *delta* peaks at even harmonics of Ω, which follows from symmetry considerations [5(d)].

Away from the peaks the correction given by Eq. (11) is always small compared to Eq. (8). The *large-frequency limitation* to the validity of the latter comes from the intrawell dynamics which was disregarded so far. As long as we are considering frequencies $\omega \ll 1$ the intrawell input to the spectral density is given by a frequency independent constant of the order of D. Comparing this with Eq. (8) one obtains the "window" where the spectral density can be approximated by the sum of Eqs. (8) and (11), i.e.,

$$(D/A)^{1/2} < \omega/\omega_K^{\max} < (\omega_K^{\max})^{-1/2}(AD)^{-1/4}. \tag{12}$$

The width of this window is asymptotically large. For larger frequencies (though still for $\omega \ll 1$) the values of $\bar{S}(\omega)$ will approach the "intrawell constant," $\sim D$, but the peaks will protrude over this value provided the driving amplitude A exceeds an asymptotically small threshold $(\omega_K^{\max})^{2/9}D^{1/3}$.

To illustrate and verify the analytical results we performed numerical studies of diffusion in a modulated quartic potential given by Eq. (9). To solve the corresponding Fokker-Planck equation a matrix continued fraction method [5(d),14] was employed. Characteristic results are depicted in Fig. 1. We used $A=0.121$, $D=\frac{1}{70}$ so that the maximal Kramers frequency was of the order of 10^{-4}. Except for very low and very high frequencies, the spectral density (dips or peaks disregarded) decays like a power law for more than 3 decades with a coefficient which is practically independent of Ω. For the driving frequencies $\Omega=10^{-5}$ and 10^{-4} one observes distinct dips at the second harmonic and also tiny dips at the fourth harmonic (see the lower inset). At higher frequencies the dips are "pierced" by narrow peaks with the height which is approximately independent of Ω. The structure of a characteristic peak is shown in the upper inset. Peaks of approximately the same width and height were also observed at higher frequency $\Omega=10^{-1}$ (not shown in the figure) where, otherwise, the spectral density behaves like a constant. At very low frequencies $\Omega=10^{-6}$ which is much smaller than the Kramers frequency, the dips vanish, so that the spectral density becomes a monotonous curve; cf. the dot-dashed line in Fig. 1.

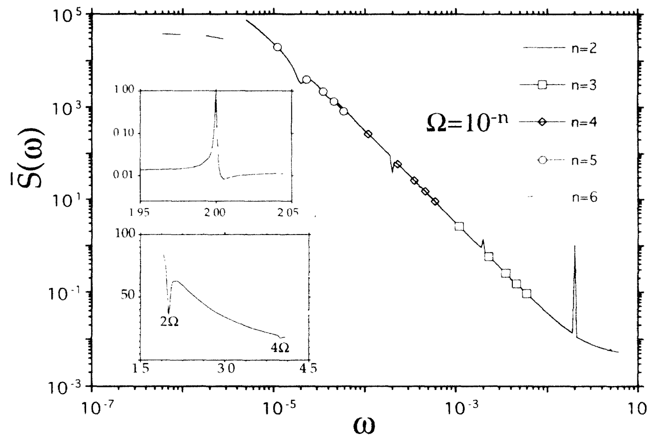

FIG. 1. Decay of the spectral density. Note the overlap of curves in the regions with no dips or peaks for different driving frequencies Ω. The insets show the spectral density as a function of reduced frequency ω/Ω in the vicinity of the dips for $\Omega=10^{-4}$ and the peak, respectively. The lower inset corresponds to $\Omega=10^{-4}$ and the upper to $\Omega=10^{-2}$. Other parameters are given in the text.

Finally, we note that similar curves with dips at the second and fourth harmonics were observed in analog simulations [3(b),10,12]. By replotting the data of these studies (especially of Ref. [10]) in a double logarithmic scale one observes that on average the spectral density decays practically like const$\times\omega^{-2}$, with the constant being independent of the driving frequency, Ω. The latter could be checked by comparing data of Figs. 3(a) and 3(b) of Ref. [10] at selected ω values, e.g., $\omega=0.1$ kHz. In accord with our analytical prediction, the dips vanish when the driving frequency is substantially lowered [10]. The peaks at even harmonics were not observed, except for the experiment with a $1/f$ noise input [12]. Most likely, colored noise can broaden the peaks, and thereby facilitates their experimental detection. Recall that in the white noise situation the broadening of peaks requires larger noise intensities. In this case the "background" spectral density given by either Eq. (8) or by the intrawell constant D increases, so that the very existence of the peaks is endangered.

We thus conclude that in the weak-noise limit there exists a broad window in the frequency spectrum with the following unusual behavior of the spectral density. On average, it decays like a power law and is insensitive to the values of the driving frequency. The latter, however, finds a way to show itself in a resonancelike manner by "piercing" the universal power law by dips or peaks at lower even harmonics. Analytically (and numerically) peaks are also expected beyond the region of the power law decay, but at present their experimental observation at higher frequencies may be problematic, at least for the white-noise case.

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[1] R. Benzi, A. Sutera, and A. Vulpani, J. Phys. A **14**, L453 (1981); C. Nicolis, J. Stat. Phys. **70**, 3 (1993).
[2] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); G. Vemuri and R. Roy, Phys. Rev. A **39**, 4668 (1989).
[3] (a) J. K. Douglass, L. Wilkens, E. Pantazelou, and F. Moss, Nature (London) **365**, 337 (1993); (b) A. Bulsara, E. Jacobs, T. Zhou, F. Moss, and L. Kiss, J. Theor. Biol. **152**, 531 (1991).
[4] See the papers in the *Proceedings of Stochastic Resonance*, edited by F. Moss, A. Bulsara, and M. Shlesinger [J. Stat. Phys. **70**, 3–514 (1993)]; and the recent review, P. Jung, Phys. Rep. **234**, 175 (1993).
[5] (a) B. McNamarra and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989); (b) R. Fox, *ibid.* **39**, 4148 (1989); (c) C. Prescilla, F. Marchesoni, and L. Gammatoni, *ibid.* **40**, 2105 (1989); (d) P. Jung and P. Hanggi, Europhys. Lett. **8**, 505 (1989); Phys. Rev. A **41**, 2977 (1990); **44**, 8032 (1991); (e) R. F. Fox and Y. N. Lu, Phys. Rev. E **48**, 3390 (1993).
[6] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).
[7] G. Hu, H. Haken, and C. Z. Ning, Phys. Lett. A **172**, 21 (1992).
[8] M. I. Dykman, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, Phys. Rev. E **47**, 1629 (1993).
[9] L. Gammatoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).
[10] T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).
[11] F. Moss, Ber. Bungsenges. Phys. Chem. **95**, 303 (1991).
[12] L. Kiss, Z. Gingl, Z. Marton, J. Kertesz, F. Moss, G. Schmera, and A. Bulzara, J. Stat. Phys. **70**, 451 (1993).
[13] F. Moss and L. Kiss (private communication).
[14] H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1984).