Controlling Spatiotemporal Chaos

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A method for controlling spatiotemporal chaos in certain classes of spatially extended systems is proposed. In these systems, unstable defects emit convectively unstable waves which subsequently break and may nucleate new defects. Control is achieved via the stabilization of one such active wave source, which then sweeps all of the chaotic fluctuations to the system boundaries. This method is applied to the one- and two-dimensional complex Ginzburg-Landau equation, and to a recent model of spiral breakup in excitable media.

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The control of the chaotic behavior of dynamical systems by weak systematic disturbances is a challenging problem. Significant progress has recently been achieved in controlling chaos in the systems with few degrees of freedom. Small carefully chosen perturbations of an arbitrary parameter of the system can confine the system near a desired unstable fixed point or unstable periodic orbit (see, e.g., [1]). This method has been successively tested in a number of experiments [2].

The control of spatiotemporal chaos leading up to the control of turbulence is a much more complicated but also a much more important problem. Since in spatially extended systems there is typically a very large number of unstable degrees of freedom, a naive point of view is that one always needs distributed control or at least some dense lattice of controlling nodes. The suppression of turbulent fluctuations via weak control at a single point seems to be an obviously impossible goal.

In this Letter we show that there exists a very important subclass of spatiotemporal chaotic behavior for which localized control is indeed realizable. In these systems, the mechanism of spatiotemporal chaos relies on the existence of unstable topological defects which nucleate spontaneously from an initially disordered state. These defects act as active sources of traveling waves, which themselves are convectively unstable. As the defect fluctuates, it drives the unstable mode of the emitted waves causing them to break. This then can lead directly either to phase turbulence or to nucleation of new defects; in both cases, the system never settles into a simple pattern either spatially or temporally. This type of behavior has been seen in simulations of the complex Ginzburg-Landau equation in one and two dimensions [3,4] and in a reaction-diffusion model of excitable CO catalysis [5]. Somewhat similar effects have been observed for a model of waves in cardiac tissue [6].

The basic idea of our method can be stated as follows. If one can stabilize one such wave-generating defect, the outgoing waves will sweep all other fluctuations to the system boundary. Hence, the extended system can be synchronized over lengths $\xi \sim \ln(\sigma)/\nu$, where σ is the

residual noise level and ν is a convective spatial growth rate. Since these defects are exact solutions (albeit unstable) of the equations, this stabilization can be accomplished by weak perturbations applied near the defect core. We will see explicitly how this works in the following.

We note in passing that the control of strongly anisotropic one-dimensional open-flow systems was considered recently in Ref. [7]. In that case control of the spatially homogeneous state is straightforward: standard techniques can be applied at one space location, and flow will propagate this state throughout the system. Here we are concerned with isotropic systems for which this method would not work.

We start with the complex Ginzburg-Landau equation (CGLE) [8] which is a model for any media near the threshold of a long-wave oscillatory instability. Typical systems modeled by this equation include transversely extended lasers [9,10] and electrohydrodynamic convection in liquid crystals [11]. The CGLE takes the form

$$\partial_t A = A + (1 + ib)\Delta A - (1 + ic)|A|^2 A \qquad (1)$$

and exhibits spatiotemporal chaos in a wide range of the parameters b, c. Various types of defects have been found as localized solutions of the CGLE. These defects are the Nozaki-Bekki holes [12] in 1D, spirals in 2D [8], and scroll waves (or vortex lines) in 3D. Stability limits of topological defects, ranges of parameters corresponding to the convective instability of plane waves, and the borders of spatiotemporal chaos in the 1D and 2D versions of CGLE have been studied in Ref. [4]. Following the general discussion given above, we work in a parameter range where waves emitted by defects are convectively unstable and defects themselves are also unstable due to a core instability [13,14].

In the 1D case, Nozaki-Bekki holes are stable only within a narrow band the in b-c plane [13,15]; otherwise, there is growth of some core-localized mode. In order to suppress turbulence in CGLE we only need to stabilize one hole in the bulk of the system by suppressing the core instability. This can be achieved adding to the right

hand side of Eq. (1) the term $\mu f(r)$, where μ is a complex number and $f(r)$ is an arbitrary localized form factor; in our simulations, we used $f(r) = 1/\cosh(\alpha r)$. In the linear approximation the perturbed motionless hole can be written in the form [13]

$$A = \left[\sqrt{1 - k^2} \tanh\left(\frac{x - x_0}{p}\right) + iBW(x - x_0) \right]$$

$$\times \exp\left[ikp \ln\cosh\left(\frac{x - x_0}{p}\right) + i\omega t \right], \qquad (2)$$

where $k(b, c)$ is the asymptotic wave number, $\omega = -c(1 - k^2) - bk^2$ is the frequency, and $p(b, c)$ is the width. The coefficient B is the amplitude of the unstable core mode which has a functional form denoted by W [for definiteness $W(0) = 1$], and is characterized by the growth rate λ. In the presence of the control scheme, the mode amplitude is governed by the equation

$$B_t = \lambda B - \delta|\mu| \sin[\arg\mu - \arg(A(x_0))], \qquad (3)$$

where δ characterizes the response of the core mode to the control. This equation is obtained by projecting the CGLE onto the single mode subspace and using the fact that B and λ are real. We instantaneously adjust the phase of μ to satisfy $\arg\mu - \arg A(x_0) = \pi/2$ and also add a dynamic equation for $|\mu|$,

$$|\mu|_t = \gamma_1|\mu| + \gamma_2 B. \qquad (4)$$

Note that the growth rate λ typically does not exceed 1; also, the value of the constant δ can be calculated accurately using perturbation theory [13,15], but for our purposes the rough estimate $\delta \approx 1$ is good enough. Therefore, one has the not very stringent conditions for the coefficients $\gamma_2 < -\lambda^2/\delta$, $\gamma_1 < -\lambda$ in order to make the coupled system stable.

To verify this scheme, we performed numerical simulations with Eq. (1) in one dimension. We used a high-resolution implicit split-step method based on the fast Fourier transform, with (typically) 1024 collocation points. The time step was chosen to be no larger than 0.05. The simulations also were checked by doubling the space discretization. The results appear to be insensitive to the resolution of the numerical scheme.
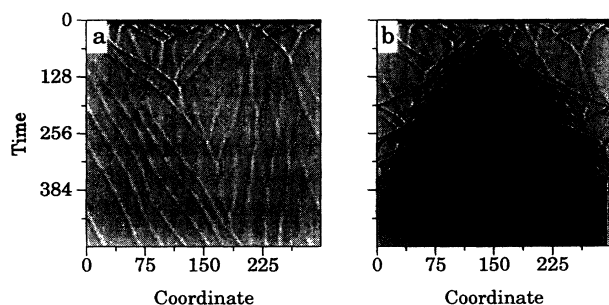
The results are presented in Figs. 1 and 2. We considered the range of phase turbulence [3], $b = -2$, c between 0.5 and 1.1, where the spatial-temporal chaos is characterized by strong phase fluctuations and almost constant amplitude. We chose the length $l = 300$ and periodic boundary conditions. Eventually, the boundary conditions are not too important because the stabilized defect breaks the symmetry of the system by emitting waves outward. In the periodic boundary conditions emitted waves collide approximately in the middle, forming shocks, whereas in other boundary conditions (e.g., no-flux) the shocks are typically formed near the edges. A typical spatiotemporal chaotic pattern as it evolves from small amplitude noise is shown in Fig. 1(a). As we switch on the control in the middle of the system, a hole is nucleated. Waves emitted by this hole sweep away the turbulent fluctuations toward the shock, and one eventually has a perfectly synchronized steady pattern [see Fig. 1(b)]. Clearly, the maximum domain which can be synchronized here exceeds the system size. The time dependence of the controller variable μ and the amplitude of the minima A is given in Fig. 2. The level of applied control in the final state is very small, $|\mu| \sim 10^{-5} - 10^{-6}$, depending on the numerical precision of the simulations.

We also performed simulations to demonstrate the control of already developed chaos. Formally speaking our considerations based on linear stability analysis fail because one has no well-established holes in the turbulent regime. However, the control creates a hole relatively quickly; in our simulations it took about 15 dimensionless time units. If, however, the level of control is restricted to below some value μ_c, the probability of "locking" decreases. Moreover, in the range of phase turbulence small values of $|A|$ are very unlikely, and one needs the value of μ_c to be above some threshold μ_0 in order to force $|A|$

FIG. 1. Evolution of $|A|$ (black $\equiv$ 0, white $\equiv$ 1) in 1D CGLE with $b = -2$, $c = 0.8$, and $L = 300$: (a) no control; (b) the hole in the middle of the system is controlled since $t = 20$, $\gamma_1 = \gamma_2 = -5$.
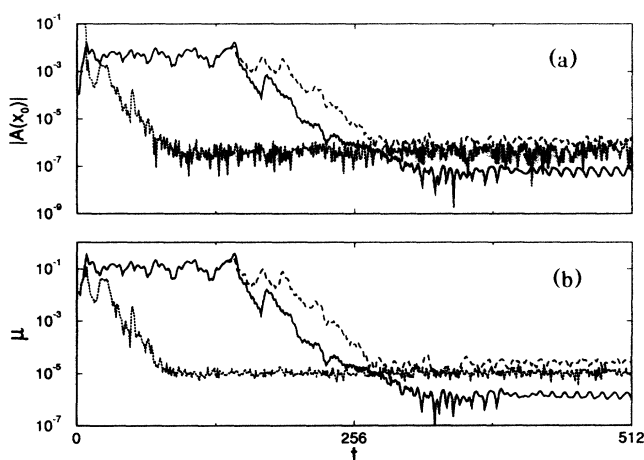

FIG. 2. Time dependencies of the magnitude of $|A|$ at the control point (a) and the control level (b); dashed line, single precision in 1D; solid line, double precision in 1D; dotted line, single precision in 2D.

to actually vanish at the point of control. In the case of $\mu_c \leq \mu_0$ the system becomes fixed at some state, characterized by $\mu = \mu_c$ and $|A(r_0)| = A(0) > 0$. This state also serves as a source of waves and may stabilize the system. In the region of amplitude turbulence one may expect the locking even for arbitrary small μ_c, because the probability that $|A|$ is close to zero is finite, but in the 1D CGLE the range of convectively unstable amplitude turbulence is very narrow.

The situation is similar in 2D. Here the defects are sources of spiral waves

$$A(r,t) = [F(r) + BW(r,\theta)] \exp\{i[\omega t + m\theta + \psi(r)]\}, \quad (5)$$

where r, θ are the polar coordinates with the origin at point r_0, $m = \pm 1$ is the topological charge, and F, ψ are real functions with the asymptotic behavior $F(0) = \psi(0) = 0, F(r)^2 \to 1 - k^2, \psi \to kr$ for $r \to \infty$. B is the complex amplitude of the (unstable) core mode W. When $B = 0$, we have an exact spiral solution with the origin at $r = r_0$. For topological reasons, the growing mode of the core instability does not destroy the spiral, but only shifts its core away from $r = r_0$. Here $\arg B$ characterizes the polar angle of the core of the shifted spiral with respect to the point r_0. Numerical simulations show that without control this instability does not saturate in the small amplitude meandering of the spiral, but destroys the spatial coherence completely and produces extensive spatiotemporal chaos [14]. Again one can find an equation governing the evolution of B. Instead of Eq. (3) one has

$$|B|_t = \mathrm{Re}\lambda|B| - |\delta||\mu| \cos[\arg\mu + \phi - \arg(A(r_0))]. \quad (6)$$

The control is achieved by forcing the spiral to drift towards the point r_0 so as to diminish the value of $|B|$. The values of $\lambda, \delta \equiv |\delta|e^{i\phi}$ can be evaluated using the results of Ref. [16]. In practice, $|\lambda| \simeq 0.1$, $|\delta| \simeq 1$, and we choose $\arg\mu \simeq \arg A - \phi$.

The parameters of our 2D simulations were $b = 14.285, c = -0.6$, which fall in the region of spiral intermittency [14]. We now used either 128×128 or 256×256 collocation points. In the absence of control one has bursts of turbulence separated by the nucleation of well defined spirals [14]. It turns out that it is difficult to achieve an effective control starting immediately from small amplitude noise. Any defect which one attempts to control attracts a neighboring defect of the opposite sign, and they form a state with zero topological charge. This state cannot be continuously transformed to a single defect state. However, we note that after some transient behavior large spirals (unstable) are formed. These spirals emit waves which screen the core from perturbations due to other defects. Applying control in the vicinity of the core of a spontaneously nucleated spiral one can easily obtain the desired locking. Because this procedure is rather time consuming, we shortened the expectation time of "big" spiral formation by introducing initial conditions in the form of small amplitude noise plus a small

amplitude "vortex seed," chosen in order to satisfy the requisite topological condition. Once this is done, the control suppresses the core instability and eventually produces a steady synchronized pattern similar to that seen in 1D. The temporal evolution of the field (at the control point) and of the control strength was given in Fig. 2.

In a recent set of experiments [17] on the catalysis of CO on platinum single crystal surfaces, spatiotemporal chaotic states were observed. Bär and Eiswirth introduced a set of modified FitzHugh-Nagumo equations to model this behavior [5],

$$\partial_t u = \nabla^2 u + \tilde{f}(u,v)/\epsilon, \quad (7)$$

$$\partial_t v = D_v \nabla^2 v + \tilde{g}(u) - v, \quad (8)$$

with the functions $\tilde{f} = -u(u-1)[u - (v+b)/a]$, and $\tilde{g} = 0$ for $u < 1/3$, $1 - 6.75u(u-1)^2$ for $1/3 < u < 1$, and $\tilde{g} = 1$ for $u > 1$. This chaotic behavior does not occur in more traditional excitable media such as the Belusov-Zhabotinsky reaction [18]. As in other such reaction-diffusion systems, there exist spiral solutions; in a wide range of the parameters a, b the spiral core meanders [19]. The chaotic state is due to the fact that this meandering exists in the same parameter range as a convective instability of the emitted waves. The meandering excites the unstable mode, the spiral arms break, and the system becomes disordered. A similar behavior seems to occur in models of wave propagation in cardiac tissue [6], due to the coexistence of spiral meandering and an (almost) period-doubling instability of the plane wave state [20]. We should note though that in the latter case, the instability can cause spatiotemporal chaos even in the absence of meandering, in a parameter range where the emitted waves become absolutely unstable.

We now add a term $-\mu f(r - r_0)$ to the v field equation. For this system, the control μ was governed by the equation

$$\partial_t \mu = \gamma_1 \mu + \gamma_2 [v(r_0) - v_0]. \quad (9)$$

Here v_0 corresponds to the value of slow field at the center of the spiral. The control is achieved due to the fact that exactly at the spiral center $v(r_0)$ is constant whereas any deviation from the center results in periodic time dependence of $v(r_0)$. The resultant periodic oscillation of μ forces the spiral center to drift towards the point of control. Eventually the center is pinned at the control site, and the spiral emits the waves which suppress the turbulence throughout the system (see Fig. 3). The simulations for this case were carried out by modifying the v equation in the program EZ-SPIRAL by Barkley [21]. In contrast to the case of the CGLE, the exact value of v_0 is not known, and any mismatch in v_0 leads to a nonvanishing controller amplitude in the final stabilized state. In order to overcome this difficulty, we started from some arbitrary value of v_0 providing stable locking of the spiral [22]. Then we gradually decreased the value of v_0 achieving as low as possible level of μ. In practice, the minimal value of μ depends on α and for $\alpha = 1$, $\mu \simeq 0.19$;
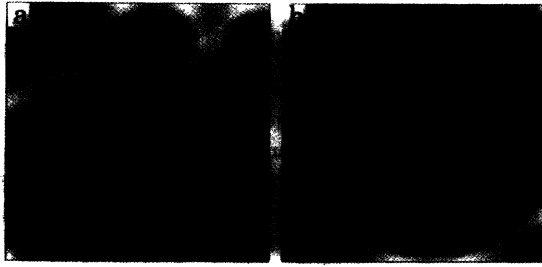
FIG. 3. Typical pattern of the slow variable v in the model (7) and (8) without (a) and with (b) control ($\alpha = 2$, $v_0 = 0.4$); parameters of the simulation: $a = 0.84$, $b = 0.07$, $\epsilon = 0.08$, and $D_v = 0.1$; system size is 30.

changing the value of v_0 past this point leads to a reemergence of the original instability. This did not happen in the CGLE and may reflect the fact that a pinned spiral does not always smoothly return to the unpinned case as the pinning strength decreases to zero. This behavior is illustrated in Fig. 4.

In this Letter we suggested an algorithm for controlling spatiotemporal chaos in isotropic continuous convectively unstable media. Possible applications of our framework include the stabilization of extended laser systems and the prevention of the transition from ventricular tachycardia (due to spiral-like rotating waves) to fibrillation. An even more challenging goal would be to control chaos in an absolutely unstable range. Simulations show that in this case the core of the defect can also be successfully locked; however, outgoing waves do not eliminate growing disturbances. At present, we know of no way to obviate the need for having a (stabilized, if necessary) array of active sources.
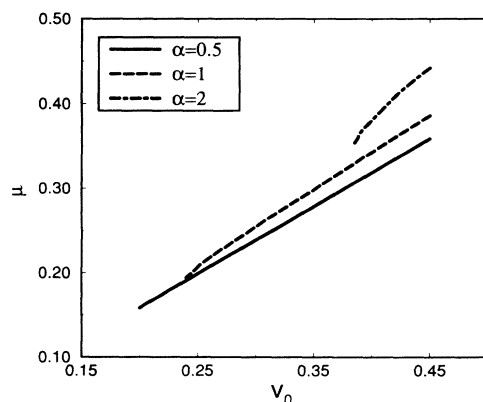
FIG. 4. Control amplitude versus v_0 for different values of the form-factor width α. Control fails to the left of the graphs.

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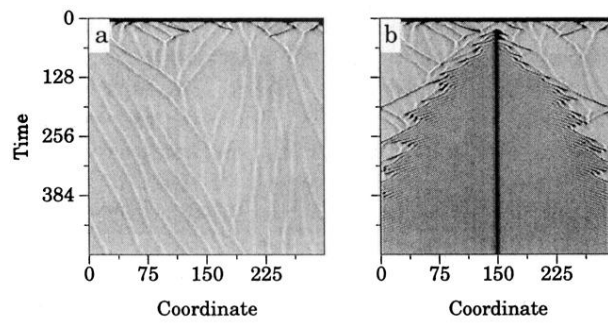
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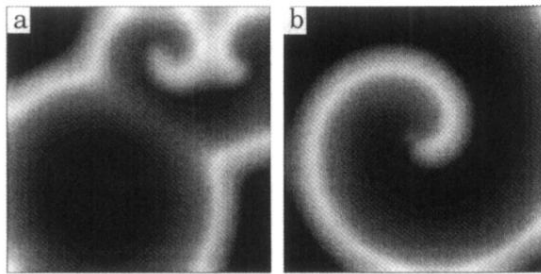
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