Energy versus Topology: Competing Defect Structures in 2D Complex Vector Field

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The topology of minimal orbits of the energy functional of a complex 2D vector field depends on the sign of nonlinear terms breaking the SU(2) symmetry, giving rise to either linearly or circularly polarized states (LP and CP) which possess different sets of defects. The CP vortices have two alternative core structures, with either vanishing amplitude or reversed polarization in the inner core. In the LP state, there are two distinct topological charges. Vortices carrying two half-unit charges have a circularly polarized core. Unit-charged vortices have a core with a vanishing amplitude, and may suffer a core instability splitting into a pair of half-unit-charged vortices.

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The problem of nonlinear dynamics of a 2D complex vector field arises most naturally in the context of nonlinear optics where the order parameter is the envelope of the electric field of a polarized wave in the plane normal to the direction of propagation. Recently, Gil [1] derived the Ginzburg-Landau (GL) equation with complex coefficients for a 2D complex vector order parameter as a normal form equation near the lasing threshold. A 2D complex vector field can be also viewed as a two-dimensional analog of the order parameter in the superfluid ^{3}He [2]. The model with a complex order parameter has been used for the description of the superconducting transition in UPt_3 [3]. Another related system is coupled nonlinear Schrödinger (NLS) equations appearing in a number of applications to nonlinear waves in fluid mechanics and plasma theory [4]. Although the latter system is not transparently geometric, it can be derived from the vector equation in a straightforward way, as shown below.

The subject of this Letter is the structure of defects in the 2D complex vector field. The theorist's interest to this problem may stem from the fact that, on the one hand, it is not altogether trivial topologically, and, on the other hand, it is sufficiently simple to obtain defect solutions in an explicit form, and to observe directly how energy considerations modify predictions of topological theory.

Further on, I shall restrict to equations with real coefficients. The stationary states of a 2D complex vector field are defined then as extrema of the energy functional

$$\mathcal{L}(\mathbf{u}) = \int \left[\partial_j \mathbf{u}\cdot\partial^j \mathbf{u}^* + V(\mathbf{u})\right] d^2\mathbf{x}. \qquad (1)$$

Here the asterisk denotes the complex conjugate, and summation over repeated coordinate indices $j = 1,2$ is presumed. The vector products in $\mathcal{L}$ are taken in such a way that the vector and gradient components do not mix. The simplest form of a potential that possesses required symmetries to spatial rotations and phase translations but breaks the maximal SU(2) symmetry is

$$V(\mathbf{u}) = \tfrac{1}{2}\left[(1-\mathbf{u}\cdot\mathbf{u}^*)^2 + \gamma(\mathbf{u}\cdot\mathbf{u})(\mathbf{u}^*\cdot\mathbf{u}^*)\right]. \qquad (2)$$

Stationary states verify the Euler-Lagrange equation

$$\nabla^2\mathbf{u} + (1-\mathbf{u}\cdot\mathbf{u}^*)\mathbf{u} - \gamma(\mathbf{u}\cdot\mathbf{u})\mathbf{u}^* = 0. \qquad (3)$$

Transforming to the null basis for the complex field $\mathbf{u} = u_+\mathbf{U} + u_-\mathbf{U}^*$, where the vector $\mathbf{U}$ satisfies the normalization conditions $\mathbf{U}\cdot\mathbf{U} = 0$, $\mathbf{U}^*\cdot\mathbf{U} = 1$, brings (3) to an alternative form

$$\nabla^2 u_\pm + \left[1 - |u_\pm|^2 - (1+2\gamma)|u_\mp|^2\right] u_\pm = 0. \qquad (4)$$

In the optical context, the basis vectors $\mathbf{U}, \mathbf{U}^*$ correspond to circularly polarized states of the opposite sense. At the same time, Eq. (4) is just a time-independent form of coupled NLS, which seems to have no geometric meaning, and is usually associated with counterpropagating waves [4].

Further on I shall use a 4D polar parametrization of the 2D complex vector field

$$u_+ = \rho\cos\frac{\theta}{2}\,\exp\frac{i}{2}(\psi+\chi),$$
$$u_- = \rho\sin\frac{\theta}{2}\,\exp\frac{i}{2}(\psi-\chi). \qquad (5)$$

The angles θ, χ can be identified as the polar and azimuthal angles parametrizing a sphere in the 3D space spanned by the Stokes parameters $z = |u_+|^2 - |u_-|^2$, $x + iy = 2u_+u_-^*$. The advantage of this parametrization lies in a simple form of the potential that makes the location of its minima transparent:

$$V(\mathbf{u}) = \tfrac{1}{2}[(1-\rho^2)^2 + \gamma\rho^4\sin^2\theta]. \qquad (6)$$

At $\gamma > 0$, the minimum of the potential (6) is achieved at one of the poles $\theta = 0$ or $\theta = \pi$, i.e., in a circularly polarized (CP) state. At $\gamma < 0$, the potential (6) is at minimum at the equator $\theta = \pi/2$. This is a linearly polarized (LP) state characterized by the parallel orientation of the real and imaginary parts of $\mathbf{u}$ [5].

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Because of the symmetries of the problem, the minima of the potential (2) or (6) are continuously degenerate, and are achieved at a certain *minimal orbit* that is defined by the action of the symmetry group of the potential. The residual symmetry corresponds to the *little group* of the minimal orbit. The character and stability of defects are determined in a standard way by the topology of the little group [6].

Defects in a more general system with complex coefficients were classified by Gil [1]. The topology is not influenced by the fact that coefficients are real, and a variational structure exists, but Gil's description of the topology in a more complex LP case is not precise, and will be corrected below.

At $\gamma = 0$, SU(2) is the symmetry group of the Lagrangian; the minimal orbit is the entire 3-sphere $\rho = 1$, and, as $\mathrm{SU}(2) = S_3$ is simply connected, there are no topological defects. The minimal orbits in the CP and LP states have different topology, and we shall study the structure of defects separately for both cases.

(a) Defects on CP background.—The little group of the CP state is O(2): the group of planar rotations and reflections. Its topology is that of two disconnected circles, $\mathrm{U}(1)\times Z_2$. There are two one-parametric families of "left" and "right" stationary CP states:

$$\text{CPL:}\ \ \mathbf{u} = \mathbf{U}\rho e^{i\phi_+},\ \ \text{CPR:}\ \ \mathbf{u} = \mathbf{U}^*\rho e^{i\phi_-}, \tag{7}$$

where $\phi_\pm = (\psi \pm \chi)/2$, respectively, at $\theta = 0$ (CPL) and $\theta = \pi$ (CPR). Stable topological defects are vortices $\mathrm{CPR}^\pm$ and $\mathrm{CPL}^\pm$ with $\phi_\pm$ rotating by $\pm 2\pi$, and kinks with θ rotating by π which separate domains with the opposite sense of polarization.

In the inner core of the vortices, deviation from the minimal stratum is necessitated by the need to relax stresses due to the rotation of phases. As it follows from the form of the potential (6), this can be done, generally, in two ways: either by depressing the modulus ρ or by pulling θ to one of the poles of the Stokes sphere [7]. In the former case, the structure of the defect core is the same as in a well-known scalar case, viz. $\theta = 0, \phi_\pm = \phi, \rho = \rho_0(r)$, where r, ϕ are polar coordinates, and the function $\rho_0(r)$ verifies

$$\rho_0'' + r^{-1}\rho_0' + (1 - r^{-2} - \rho_0^2)\rho_0 = 0 \tag{8}$$

with the boundary conditions $\rho_0(0) = 0$, $\rho_0(\infty) = 1$. Since the amplitude vanishes at the origin, it is appropriate to call this a *punched core* structure.

An alternative way of regularizing the solution can be interpreted as nucleation of a state with the opposite sense of circular polarization and a nontopological phase in the vortex core, and the resulting dissolution of the vortex into a circular kink. At $\gamma = 0$, the kink would spread out indefinitely, and the circulation disappear; this is, indeed, the surgery eliminating the defect on the SU(2) group space. At $\gamma > 0$, the energetic costs of stretching the kink are prohibitive, and deviations from the stratum $\theta = \pi$ should be confined to the vortex core. The result is the *repolarized core* structure, alternative to that of Eq. (8). For the CPR^+ vortex with $\theta = \pi$ and $\phi_- = (\psi - \chi)/2$ rotating by 2π, this structure verifies the equations

$$\begin{gathered}\rho'' + \frac{1}{r}\rho' - \frac{1}{4}\rho(\theta')^2 - \frac{1}{r^2}\rho\sin^2\frac{\theta}{2} + \rho \\ -\rho^3\left(1 + \gamma\sin^2\theta\right) = 0, \\ \theta'' + \frac{1}{r}\theta' - \frac{2}{\rho}\rho'\theta' - \frac{1}{r^2}\sin\theta - \gamma\rho^2\sin 2\theta = 0.\end{gathered} \tag{9}$$

The asymptotic conditions are $\rho(r) \to 1$, $\theta(r) \to \pi$ at $r \to \infty$. Near the origin, the singularity is cancelled when θ approaches zero linearly ($\theta = ar$ at $r \to 0$) with any slope a, whereas ρ remains finite and $\rho'(0) = 0$. The constant a, as well as the value of $\rho(0)$, should be obtained by matching to the asymptotic conditions when Eq. (9) is integrated numerically.

The core structure is particularly simple at $|\gamma| \ll 1$. Then $\rho = 1 - O(\gamma)$, and rescaling the radial coordinate $r \to r/\sqrt{\gamma}$ yields, in the leading order, the equation of θ

$$\theta'' + \frac{1}{r}\theta' \pm \frac{1}{r^2}\sin\theta - \sin 2\theta = 0. \tag{10}$$

The solution, that can be obtained numerically, exhibits an inflection point within a linearly polarized transition belt. The repolarized core is stretched at $\gamma \ll 1$ to the size of $O(\gamma^{-1/2})$, and the gain in energy relative to the compact punched core structure is of $O(\ln\gamma)$, but this advantage is apt to disappear at larger values of γ.

At $\gamma \gg 1$, the loss of linear polarization must be confined to an inner core region with a radius of $O(\gamma^{-1/2})$, and the structure in the outer core region with an $O(1)$ radius coincides in the leading order with the punched core solution; thus, the distinction between both alternative structures disappears in this limit. The contrast between both structures is most sharp in the opposite limit $\gamma \ll 1$. Vortices with a repolarized core dissolve gradually as their core radius diverges at $\gamma \to 0$. On the contrary, the punched core structure remains compact even at $\gamma = 0$, and, since all defects become unstable at $\gamma = 0$ when the SU(2) symmetry is restored, it must suffer a sudden instability as γ decreases.

The instability threshold can be estimated by directly computing second variations of the energy functional (1). The most dangerous trial variation corresponds to the excitation of the state with the opposite polarization and a nontopological phase, that can eventually result in transition to the repolarized core structure. If, say, $\mathbf{u} = \rho(r)e^{i\phi}\mathbf{U} + \widetilde{\rho}(r)\mathbf{U}^*$, the potential energy is expressed as $-2\pi\int_0^\infty \widetilde{\rho}(r)^2\left[1 - (1 + 2\gamma)\rho_0^2(r)\right] r\,dr$. Since the integrand changes sign only sufficiently far from the origin, one can choose a trial function $\widetilde{\rho}(r)$ smoothly dependent on the radius within some circle with the radius r_0, and vanishing outside. The value of r_0 should be chosen in such a way that the positive contribution to the kinetic energy in Eq. (1) would not overweigh the decrease in potential energy. One can obtain a lower estimate of the stability boundary numerically by setting $\widetilde{\rho}(r) = 1 - r/r_0$ at $r < r_0$, $\widetilde{\rho}(r) = 0$ at $r > r_0$. Choosing $r_0 \approx 4.5$ maximizes the value of γ at which the second variations of the energy functional vanishes. Computations using the numerical solution of Eq. (8) show that the punched core structure is unstable at $\gamma < 0.17$. This lower estimate can be only slightly improved by choosing the trial variation in the form $\widetilde{\rho}(r) = 1 - (r/r_0)^n$ with $n < 1$.

(b) Defects on LP background.—The two-parametric family of stationary LP states, parametrized by two independent phases ψ and χ, can be presented as

$$\mathbf{u} = \rho e^{i\psi/2} \begin{vmatrix} \cos\left(\dfrac{\chi}{2} + \dfrac{\pi}{4}\right) \\ \sin\left(\dfrac{\chi}{2} + \dfrac{\pi}{4}\right) \end{vmatrix} . \tag{11}$$

Since rotating ψ by 2π is equivalent to rotating χ by the same angle, the topology of the LP minimal stratum at $\gamma < 0$ is that of a torus *with opposite points identified*: $U(1) \times U(1)/Z_2$. There are two topological charges, to be called *argument* and *director* charges. The topological defects are vortices $\mathrm{LP}(n,m)$ with ψ rotating by $4n\pi$ and χ rotating by $4m\pi$ where $n+m$ is an integer. The eight elementary topological defects (Fig. 1) are *argument* vortices and antivortices ($n = \pm 1$, $m = 0$), *director* vortices and antivortices ($m = \pm 1$, $n = 0$), and two kinds of *depolarized* vortices and antivortices ($m = n = \pm\frac{1}{2}$ and $m = -n = \pm\frac{1}{2}$). The reason for naming the vortices with half-unit charges in this way will become clear below. The identification of the opposite points in the group space, that makes defects with half-unit topological charges possible. was overlooked by Gil [1]. Defects with multiple argument and director charges are expected to be unstable to splitting into elementary vortices, and need not to be considered here. We shall see, however, that the same fate can await elementary vortices that seem to enjoy topological protection.

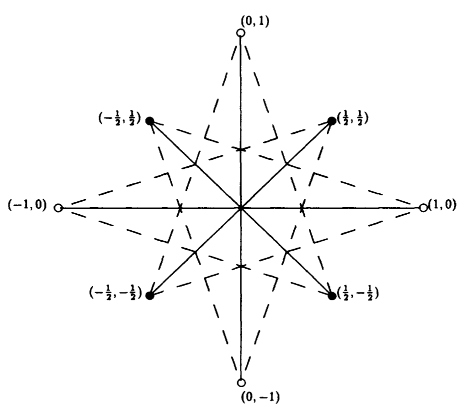

FIG. 1. Eight elementary vortices on the LP background. The abscissa and ordinate correspond to director and argument charges. Solid lines connect vortices that interact attractively and annihilate at collision. Dashed lines connect vortices that interact attractively and merge to another vortex (closest to the line) when the lines collide.

The director and argument vortices are equivalent, respectively, to rotations of a real vector or a complex scalar. In both cases, the punched core solutions (8) with $\rho_s = (1-|\gamma|)^{-1/2}\rho_0(r)$ vanishing at the center of the vortex are applicable. As in the CP state, these solutions become, however, unstable at sufficiently small $|\gamma|$.

Consider first the LP argument vortex. It can be viewed as a rotation of a single orthogonal component of the vector field if one chooses as basis vectors $\mathbf{U}_+ = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$, $\mathbf{U}_- = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$. The punched core solution Eq. (11) can be presented as $\mathbf{u}_0 = \mathbf{U}_+ \rho_s(r) e^{i\phi}$. We choose the perturbation in the form $\widetilde{\mathbf{u}} = \widetilde{\rho}\mathbf{U}_-$. The second variation of the term $|\mathbf{u}\cdot\mathbf{u}|^2$ multiplying γ is expressed as $\int \rho^2(r)\widetilde{\rho}^2 \cos 2\phi \, r\, dr\, d\phi$, and vanishes if $\widetilde{\rho}$ depends on the radius only. The remaining potential term is $-2\pi \int_0^\infty \widetilde{\rho}^2[1-\rho_s^2(r)]\, r\, dr$. Using the same estimates as for CP vortices above shows that the punched core structure is unstable at $|\gamma| < 0.254$.

The same estimate can be obtained for the director vortex by choosing a somewhat different orthogonal decomposition

$$\mathbf{u} = \mathbf{u}_0 + \widetilde{\mathbf{u}}, \quad \mathbf{u}_0 = \rho_s(r) \begin{vmatrix} \cos\phi \\ \sin\phi \end{vmatrix}, \quad \widetilde{\mathbf{u}} = i\widetilde{\rho} \begin{vmatrix} 1 \\ 0 \end{vmatrix},$$

with real $\widetilde{\rho}(r)$. The second variation of the term $|\mathbf{u}\cdot\mathbf{u}|^2$ vanishes as above, and the stability limit can be estimated exactly in the same way as for the argument vortex, yielding the identical result.

When both ψ and χ are rotating, the punched core structure is impossible, and linear polarization must be lost in the vortex core. The regularization of the vortex core is achieved when θ approaches one of the poles of the Stokes sphere. If $\psi = -\chi = \pm\phi$ rotate in the opposite senses, the radial core structure $\rho(r)$, $\theta(r)$ verifies Eq. (9), and $\theta = ar$, $\rho'(r) = 0$ at $r \to 0$. Similar equations apply when $\psi = \chi = \pm\phi$ rotate in the same sense, and $\theta = \pi - ar$ with a indefinite, $\rho'(r) = 0$ at $r \to 0$. The asymptotic conditions determining the appropriate values of a and $\rho(0)$ are identical in both cases: $\rho(r) \to (1-|\gamma|)^{-1/2}$, $\theta(r) \to \pi/2$ at $r \to \infty$. In both cases, the inner core is circularly polarized with opposite senses of polarization.

Another way to view the structure of depolarized vortices is to consider the linearly polarized state as a mixture of two circularly polarized states. The far field structure of a circularly symmetric depolarized vortex is given by $\mathbf{u} = \rho_+(r)e^{\pm i\phi}\mathbf{U} + \rho_-(r)\mathbf{U}^*$ when ψ and χ rotate in the same sense, and $\mathbf{u} = \rho_+(r)\mathbf{U} + \rho_-(r)e^{\pm i\phi}\mathbf{U}^*$ when they rotate in the opposite senses. This means that only one of the components rotates, while the other one has no angular dependence. The amplitude of the rotating component has to vanish at the origin, while the other one remains finite; accordingly, the inner core acquires the circular polarization of the nonrotating component.

For the integer-charged argument and director vortices, the stress is not relaxed at either pole of the Stokes sphere, and the punched core structure is the only possibility. The surgery eliminating these defects requires passing *both* poles consecutively. This is achieved when the vortex splits into two depolarized vortices with opposite senses of circular polarization in the core regions.

Instability of CP vortices leads just to a change of the core structure, with no change of topology. The instability of LP vortices must have more far-reaching consequences. Since LP argument and director vortices do not possess an alternative core structure, their instability should lead to splitting into a pair of depolarized vortices. While the total director and argument charges remain conserved in this decay reaction, the separation of the two splinters due to a repulsive interaction would spread the topological restructuring into a large spatial domain.

(c) Vortex interactions.—Dynamics can be defined, starting from the energy functional (1), in two contrasting ways:

$$\mathbf{u}_t = -\delta\mathcal{L}/\delta\mathbf{u}^* \quad \text{or} \quad \mathbf{u}_t = -i\delta\mathcal{L}/\delta\mathbf{u}^*. \tag{12}$$

The first option is *gradient dynamics*, whereby energy decreases monotonically in the course of evolution. The second option means that energy is *conserved* (other conserved quantities are densities and momenta of two "superfluid" components [4]). The structure of static defect solutions is identical in both cases, but the way they interact is totally different.

Far from the vortex core, deviations of ρ and θ from their constant asymptotic values corresponding to the minimum of the potential are of $O(r^{-2})$, while the gradients of the phase variables ψ and χ are of $O(r^{-1})$. If the distance between the centers of two vortices $r = \epsilon^{-1} \gg 1$ is large compared to the core size, the force setting each vortex into motion is determined, in the leading order, by the local gradient of the phase fields of the other vortex. In a diluted vortex gas, the motion is induced by the gradient of the total phase fields of all other vortices.

Equations of vortex motion are obtained, using solvability conditions of linearized equations in the core region, in the same way as for vortices in scalar fields [8], and will be considered in detail elsewhere [9]. In the case of gradient dynamics, vortices move across a weighted phase gradient, and the mobility relationship contains a usual logarithmic correction. Attractive interactions between the eight elementary vortices on the LP background are shown schematically in Fig. 1. The interaction vanishes for vortices bearing charges of different nature or for depolarized vortices with different sense of polarization of the inner core (e.g., $n = -m = \frac{1}{2}$ and $n' = m' = \frac{1}{2}$). In these cases, a weaker interaction due to induced dipole moments at vortex cores, which decays proportionally to the cube of the distance, can be detected by continuing the expansion to $O(\epsilon^3)$.

In the conservative case, the long-scale dynamics of the linearly polarized field corresponds, in the leading order, to the flow of two interpenetrating but noninteracting incompressible superfluids that advect vortices with respective charges. A general case of Eq. (3) with complex coefficients, that possesses neither gradient nor Hamiltonian structure, is far more complicated. Generally, one should expect that vortices would radiate waves with a certain wavelength, analogous to the spiral waves in the scalar GL equation with complex coefficients [10]. In addition to the usual punched core structure of a rotating spiral wave [10], one should expect also in this case formation of alternative repolarized core structures. It would be premature to discuss interaction of vortices in this setting, as even for the scalar case has been studied systematically only under very limiting conditions [11]. Only if the combination of coefficients is such that the residual wave number vanishes, the dynamics is intermediate between the gradient and conservative cases, so that vortices move at an oblique angle to the phase gradients.

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