Spectrum of the QCD Dirac Operator and Chiral Random Matrix Theory

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(Received 17 January 1994)

We argue that the spectrum of the QCD Dirac operator near zero virtuality can be described by random matrix theory. As in the case of the classical random matrix ensembles of Dyson we have three different cases: the chiral orthogonal ensemble, the chiral unitary ensemble, and the chiral symplectic ensemble. They correspond to gauge groups SU(2) in the fundamental representation, $SU(N_c)$, $N_c \geq 3$ in the fundamental representation, and non-Abelian gauge groups $SU(N_c)$ for all N_c with fermions in the adjoint representation, respectively. The joint probability density reproduces Leutwyler-Smilga sum rules.

PACS numbers: 12.38.Aw, 11.15.Pg, 11.30.Rd, 11.55.Hx

According to the Banks-Casher formula [1], the spectrum of the Dirac operator near zero virtuality is directly connected with a nonzero value of the chiral condensate, the order parameter of the chiral phase transition. This suggests that the spectrum in this region plays an important role in understanding the mechanism of chiral symmetry breaking. Recently, it was shown that a nonzero value of the chiral condensate leads to the existence of sum rules for inverse powers of the eigenvalues of the Dirac operator [2]. These sum rules are only sensitive to the spectrum near zero virtuality and can be expressed in terms of microscopic spectral correlation functions, which measure correlations on the order of a finite number of average level spacings [3,4]. As is well known from the study of chaotic systems [5,6], such correlations are independent of the details of the interactions and can be described by random matrix theory. Indeed, we [3,4] have shown that for a complex Dirac operator *all* Leutwyler-Smilga sum rules follow from a chiral random matrix theory. This led to the claim that the microscopic spectral correlation functions of the Dirac operator are universal.

More than three decades ago Dyson [7] found three distinct types of random matrix ensembles: the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE), and the Gaussian symplectic ensemble (GSE), corresponding to real, complex, and quaternion matrix elements. The random matrix theory discussed in [3,4] has complex matrix elements. For that reason it has been named the chiral unitary ensemble (chGUE). This raises the questions of what the chiral analogs of the GOE and the GSE are, and what the structure of the corresponding Dirac operator is. The answer to these questions will be given in this Letter. We will also derive the joint eigenvalue density of the random matrix ensembles and present the result for the simplest Leutwyler-Smilga sum rule.

The Euclidean Dirac operator in QCD is defined by

$$D \equiv i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu , \tag{1}$$

where A is an $SU(N_c)$ valued gauge field (N_c is the number of colors). Because this operator anticommutes with γ_5, in a chiral basis it reduces to the following block structure

$$\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} . \tag{2}$$

In general (for $N_c \geq 3$), for fermions in the fundamental representation, the matrix elements of the Dirac operator, i.e., T_{ij}, are complex. This defines the first family of Dirac operators.

However, in the case of SU(2) we have an additional symmetry [2], which is specific to this group:

$$[C^{-1}\tau_2 K, D] = 0 , \tag{3}$$

where C is the charge conjugation matrix ($\gamma_\mu^* = -C\gamma_\mu \times C^{-1}$), and K denotes the complex conjugation operator. This symmetry operator has the property that

$$(C^{-1}\tau_2 K)^2 = 1 . \tag{4}$$

As is well known from the analysis of the time-reversal operator in random matrix theory [8], this property allows us to choose a basis in which the matrix elements of the Dirac operator in (2) are real and $T^\dagger = \tilde{T}$. This provides us with the second family of Dirac operators.

The third family of gauge theories are those with fermions in the adjoint representation. The Dirac operator is given by

$$D_{kl} = i\gamma_\mu \partial_\mu \delta_{kl} + i f^{klm} \gamma_\mu A_{\mu m} , \tag{5}$$

where f^{klm} are the structure constants of the gauge group. As was noted in [2], in this case the covariant derivative is real, and the Dirac operator is invariant under charge conjugation

$$[D, C^{-1}K] = 0 . \tag{6}$$

Because $C^* C^{-1} = -1$, one can easily derive that

$$(C^{-1}K)^2 = -1 . \tag{7}$$

This implies that each eigenvalue of the Dirac operator is doubly degenerate with linearly independent eigenfunc-

0031-9007/94/72(16)/2531(3)$06.00

tions [8] given by

$$\phi_\lambda \text{ and } C^{-1}K\phi_\lambda . \tag{8}$$

As follows from a discussion by Dyson [9], in this case the Dirac operator can be diagonalized by a symplectic transformation. Or, in other words, the matrix elements of T can be regrouped into real quaternions, and $T_{ij}^\dagger = \bar{T}_{ji}$ (quaternion conjugation is denoted by a bar).

The QCD partition function for N_f flavors with masses m_f ($m_f \to 0$) in the sector with ν zero modes is defined by

$$Z_\nu^{\rm QCD} = \left\langle \prod_{f=1}^{N_f} \prod_{\lambda_n > 0} (\lambda_n^2 + m_f^2) m_f^\nu \right\rangle_{S_\nu(A)}, \tag{9}$$

where the average $\langle \cdots \rangle_{S_\nu(A)}$ is over gauge field configurations with ν fermionic zero modes weighted by the gauge field action $S_\nu(A)$. The product is over all eigenvalues of the Dirac operator. For fermions in the adjoint representation, the doubly degenerate eigenvalues count only once (Majorana fermions) [2]. Relevant observables are obtained by differentiation with respect to the masses. The distribution of the eigenvalues of the Dirac operator is induced by the fluctuations of the gauge field. It is our claim that the correlations between eigenvalues near zero virtuality, i.e., of the order of a finite number of level spacings away from zero, do not depend on details of the interaction.

The basic underlying idea of random matrix theory is that correlations between eigenvalues on the scale of one eigenvalue are only determined by the symmetries of the system and do not depend on the detailed dynamics. Therefore, the spectral density measured in units of the average spectral density near zero virtuality is a universal quantity that can be described by a random matrix theory that reflects only on the *symmetries* of the Dirac operator. The relevant random matrix theory in the sector with ν zero modes is

$$Z_{\beta,\nu} = \int \mathcal{D}T\, P_\beta(T) \prod_f^{N_f} \det \begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix}, \tag{10}$$

where T has the symmetries of the corresponding Dirac operator and the masses are in the chiral limit ($m_f \to 0$). For SU(2) in the fundamental representation T is real ($\beta = 1$), for SU(N_c), $N_c \geq 3$, the matrix T is complex ($\beta = 2$), and for fermions in the adjoint representation, T is quaternion real ($\beta = 4$). In the latter case the square root of the fermion determinant appears in (10). The matrix T is a rectangular $n \times m$ matrix with $|n - m| = \nu$ (for definiteness we take $m > n$ and $\nu \ll n$). It can be shown that the matrix in the fermion determinant in (10) has exactly ν zero eigenvalues (and $Z \sim \prod_f m_f^\nu$). The distribution function of the matrix elements $P(T)$ that is consistent with no additional information input is Gaussian [10]. In a standard normalization we choose

$$P_\beta(T) = \exp\left(-\frac{\beta n}{2\sigma^2} \sum_{k=1}^{n} \lambda_k^2 \right), \tag{11}$$

where the sum is over the nonzero eigenvalues of T. With this choice the average spectral density does not depend on β and is given by $\rho(0) = 1/\pi\sigma$. This allows us to identify $\sigma = 1/\pi\rho(0) \equiv 1/\Sigma$, where Σ denotes the chiral condensate. It should be pointed out that, as in the QCD partition function, the chiral symmetry is broken isotropically in flavor space.

In order to derive the joint eigenvalue density we use the eigenvalues and eigenangles of T as new integration variables. In each of the three cases, up to a constant, the Jacobian of this transformation is given by

$$J = \prod_{k<l} |\lambda_k^2 - \lambda_l^2|^\beta \prod_k \lambda_k^{\nu\beta + \beta - 1}. \tag{12}$$

The derivation of this result will be given elsewhere. At the moment we only remark that the total powers of λ_k can be obtained on dimensional grounds only. The joint eigenvalue density is therefore given by

$$\rho_\beta(\lambda_1, \ldots, \lambda_n) = C_{\beta,n} \prod_{k<l} |\lambda_k^2 - \lambda_l^2|^\beta \prod_k \lambda_k^{2N_f + \beta\nu + \beta - 1} \times e^{n\beta\Sigma^2/2 \sum_k \lambda_k^2}. \tag{13}$$

The normalization constant is denoted by $C_{\beta,n}$. In the case of $\beta = 4$, each of the doubly degenerate eigenvalues is counted only once. This is consistent with the fact that fermions in the adjoint representation are Majorana fermions (see [2] for a detailed discussion of this point). The simplest Leutwyler-Smilga sum rule in the sector with ν zero modes can be evaluated with the help of Selberg's integral [11] (see [12] for a discussion). The result is

$$\left\langle \frac{1}{N^2} \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} \right\rangle_{\rho_\beta} = \frac{\beta\Sigma^2}{8(\beta\nu/2 + \beta/2 + N_f - 1)}, \tag{14}$$

where the average is with respect to the spectral density $\rho_\beta(\lambda_1, \ldots, \lambda_n)$. The total number of modes is denoted by $N = m + n$. This constitutes the final result of this Letter. It agrees with sum rules obtained by Leutwyler and Smilga for $\beta = 2$ and $\beta = 4$. This sum rule can also be expressed in terms of the microscopic spectral density defined by

$$\rho_S(x) = \lim_{N \to \infty} \frac{1}{N} \rho\left(\frac{x}{N} \right), \tag{15}$$

where $\rho(\lambda)$ is obtained from (13) by integrating over all eigenvalues except one. The microscopic spectral density has been derived for $\beta = 1$ [13] and $\beta = 2$ [4], and in both cases it agrees with numerical results from simulations of gauge field configurations by a liquid of instantons [14].

In [2] sum rules were derived from the static limit of an effective field theory. The above discussed triality implies that we have three structurally different effective field theories. Two of them were analyzed in [2], and the theory for $\beta = 1$, which involves both baryons and mesons, was discussed in [15]. It would be instructive to derive (14) for $\beta = 1$ in the framework of this model.

In conclusion, we have argued that, depending on the gauge group and the representation of the fermions, the QCD Dirac operator falls into three different families: SU(2) in the fundamental representation, $SU(N_c)$, $N_c \geq 3$, in the fundamental representation, and $SU(N_c)$ in the adjoint representation. This triality corresponds to real, complex, and quaternion matrix elements. Its spectrum near zero virtuality reflects only the symmetries of the system and can be described in terms of chiral random matrix theory: the chGOE, the chGUE, and the chGSE, respectively. Sum rules obtained from general arguments based on effective field theory [2] are reproduced.

The reported work was partially supported by the U.S. DOE Grant No. DE-FG-88ER40388. I would like to thank M. A. Nowak for useful discussions and a critical reading of the manuscript. T. Schaefer is thanked for stimulating discussions and E. Shuryak for a critical reading of the manuscript. Professor H. Weidenmüller is thanked for continuous encouragement.

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