Decoherence, Chaos, and the Second Law

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(Received 8 November 1993)

Quantum wave function of a chaotic system spreads rapidly over distances on which the potential is significantly nonlinear. As a result, the effective force is no longer just a gradient of the potential, and predictions of classical and quantum dynamics begin to differ. We show how the interaction with the environment limits distances over which quantum coherence can persist, and therefore reconciles quantum dynamics with classical Hamiltonian chaos. The entropy production rate for such open chaotic systems exhibits a sharp transition between reversible and dissipative regimes, where it is set by the chaotic dynamics.

PACS numbers: 03.65.Bz, 05.40.+j, 05.45.+b

The relation between classical and quantum chaos has been always somewhat unclear [1] and, at times, even strained [2]. The cause of the difficulties can be traced to the fact that the defining characteristic of classical chaos—sensitive dependence on initial conditions—has no quantum counterpart: It is defined through the behavior of neighboring trajectories [3], a concept which is essentially alien to quantum mechanics. Moreover, when the natural language of quantum mechanics of closed systems is adopted, an analog of the exponential divergence cannot be found. This is not to deny that many interesting insights into quantum mechanics have been arrived at by studying quantized versions of classically chaotic systems [4]. These insights have typically much to do with the energy spectra, and leave the issue of the relationship between the quantum and classical evolutions largely open.

The aim of this paper is to investigate implications of the process of decoherence for quantum chaos. Decoherence is caused by the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment [5,6]. Preferred states are singled out by their stability (measured, for example, by the rate of predictability loss—the rate of entropy increase) under the joint influence of the environment and the self-Hamiltonian [7]. Thus, the strength and nature of the coupling with the environment play a crucial role in selecting preferred states, which—given the distance-dependent nature of typical interactions—explains the special function of the position observable [5,7]. Coupling with the environment also sets the decoherence time scale [8]—the time on which quantum interference between preferred states disappears [5–9]. Classicality is then an emergent property of an open quantum system. It is caused by the incessant monitoring by the environment, the state of which keeps a "running record" of the preferred observables of the evolving quantum system. For simple quantum systems the program sketched above can be carried out rigorously, and yields intuitively appealing results [5,9, 10]. For example, preferred states of an underdamped harmonic oscillator turn out to be its coherent states [10].

If decoherence does induce a transition from quantum to classical, then it should be possible to utilize it in the context of quantum chaos to establish a more straightforward correspondence between the behavior of classically chaotic systems and their quantum counterparts. With this goal in mind we will consider a chaotic system, characterized by a potential $V(x)$, coupled to an external environment. Evolution of such an open quantum system can be studied under a variety of reasonable assumptions [11–13]. Here we focus on the simplest special case, the high temperature limit of an Ohmic environment. In this case, the Wigner function of the system evolves according to [11]

$$\dot{W}=\{H,W\}_{\mathrm{PB}}+\sum_{n\geq 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\partial_x^{2n+1}V\partial_p^{2n+1}W + 2\gamma\partial_p(pW)+D\partial_{pp}^2W\,, \qquad (1)$$

where γ is the relaxation rate and the diffusion coefficient is $D=2\gamma M k_B T$ (T is the temperature of the environment and M is the mass of the system). The first term is the Poisson bracket, which generates the ordinary Liouville flow. Both the Poisson bracket and the higher derivative terms result from an expansion of the Moyal bracket, $\{H,W\}_{\mathrm{MB}}=-i\sin(i\hbar\{H,W\}_{\mathrm{PB}})/\hbar$, which generates evolution in phase space of a closed quantum system [this expansion is valid when $V(x)$ is analytic]. The last two terms in (1) arise due to the interaction with the environment. The first of them produces relaxation—exchange of energy with the reservoir—and the last one diffusion (which is responsible for the *decoherence process*).

The failure of correspondence between quantum and classical dynamics in a chaotic system is easy to understand. Quantum correction terms in (1) contain higher derivatives. As a result the Wigner function does not follow the classical Liouville flow. In a quantum analog of a classically chaotic system these nonclassical corrections rapidly gain importance: When a chaotic flow is investigated locally in the phase space, the evolution operator can be expanded in coordinates "comoving" with a reference trajectory. The pattern of flow of the neighboring trajectories is then generated by the Jacobian of the transformation. Eigenvalues of this Jacobian are known as *local Lyapunov exponents* λ_i, which must sum to zero, since the transformation preserves phase space volume.

0031-9007/94/72(16)/2508(4)$06.00

Eigenvectors define directions in the phase space along which the neighboring trajectories either only expand ($\lambda_i > 0$) or only contract ($\lambda_i < 0$) with respect to the fiducial trajectory at a rate given by the corresponding Lyapunov exponent [3]. The exponential contraction rapidly generates small scale structure in the Wigner function. Thus, the higher derivative terms are $\partial_p^n W \propto \sigma_p^{-n} W$ with $\sigma_p \propto \sigma_p(0)\exp(\lambda t)$, where λ is a Lyapunov exponent. Hence, nonclassical corrections will become important after a characteristic time t_χ which can be estimated by comparing the magnitude of the nonlinear corrections with the contribution of the Poisson bracket in Eq. (1). Defining a characteristic scale for the nonlinear terms as $\chi_n \propto (\partial_x V/\partial_x^{n+1} V)^{1/n}$ we obtain that the nth order term in (1) becomes comparable with the Poisson bracket at a time $t_\chi^{(n)}$ given by

$$t_\chi^{(n)} \propto \lambda^{-1} \ln[\chi_n \sigma_p(0)/\hbar]\,. \tag{2}$$

Below we shall see that in the presence of decoherence the regime in which the quantum corrections become important is easily avoided: Diffusive effects put a lower bound on the small scale structure which can be produced by chaotic evolution. As a result, $\sigma_p(t)$ can never become sufficiently small to result in large quantum corrections: The classical Poisson bracket is an excellent approximation of the quantum Moyal bracket for smooth Wigner functions. As we will also demonstrate, this interplay of decoherence and dynamics has important consequences for the rate of entropy production.

In our analysis, based on Eq. (1), we will neglect the relaxation term which can be made arbitrarily small without decreasing the effectiveness of the decoherence process (e.g., by letting γ approach zero while keeping D constant [8]). In this way, we will focus on the important *reversible classical limit* [7,8]. Models leading to Eq. (1) break the symmetry bewteen x and p by coupling with the environment solely through position. It is convenient (especially in the context of quantum optics, where the "rotating wave approximation" can be invoked) to use a symmetric coupling (of the form $a^\dagger b + ab^\dagger$, where a and b are the annihilation operators of the system and a mode of the environment [14]). The corresponding equation differs from (1) in the form of the diffusion which is now symmetric, $\propto D(\partial_{pp}^2 + \partial_{xx}^2)W$. We shall alternate between using this symmetric diffusion and the more exact diffusion operator of Eq. (1) in the discussion below.

To study the interplay between the evolution which classically results in an exponential divergence of neighboring trajectories (the characteristic feature of chaos) and the destruction of quantum coherence between a preferred set of states in the Hilbert space of an open system (a defining feature of decoherence), we shall use a simple unstable system that still captures the essential features we want to consider. In general, a chaotic geodesic flow pattern is locally analogous to the one occurring near a saddle point, with stable and unstable directions defined in an obvious manner. The simplest example of such a saddle point is afforded by an *unstable harmonic oscillator*. In this case, the potential is $V(x) = -\lambda x^2/2$ (λ is the Lyapunov exponent) and Eq. (1) reduces to a very simple form. We will use the lessons learned from this simple unstable system to argue that quantum corrections, which in the case of the unstable oscillator vanish identically, can be neglected whenever decoherence is effective. Our analysis will also show that three different aspects of the evolution can be identified: (i) decoherence, (ii) approximately reversible Liouville flow, and (iii) irreversible diffusion-dominated evolution.

To analyze in detail the unstable oscillator it is convenient to use contracting and expanding coordinates defined by ${}^u_v = p \mp M\lambda x$. Evolution generated by Eq. (1) causes exponential expansion in v and, without the diffusive term, it would also cause an exponential contraction in u, so that the volume in the phase space (as well as entropy) would be constant (see Fig. 1). Expansion in v would also result in an exponential decrease of gradients in that direction. Thus, after a sufficient number of e-foldings the equation governing evolution of W would be dominated by the expression

$$\dot W = \lambda(u\partial_u - v\partial_v + \tfrac{1}{2}\sigma_c^2 \partial_{uu})W\,. \tag{3}$$

The characteristic dispersion, which will play an important role below, is

$$\sigma_c^2 = 2D/\lambda\,. \tag{4}$$

The general solution of Eq. (3) can be found by noticing that the eigenfunctions of the operator appearing in its right hand side are $v^n F_m(u/\sigma_c)$, where $F_m(x) = \exp(-x^2/2)H_{m-1}(x/\sqrt{2})$ and $H_m(x)$ are Hermite polynomials. Expanding $W(u,v,t)$ in terms of these eigenfunctions [whose eigenvalue is simply $-(n+m)$], we obtain

$$W(u,v,t) = \sum_{\substack{n\ge 0\\ m\ge 1}} a_{nm}(ve^{-\lambda t})^n F_m(u) e^{-m\omega t}\,. \tag{5}$$

Therefore, the Wigner function depends on v only through the combination $v_0 = v\exp(-\lambda t)$, which is the comoving coordinate. That is, along this direction, W just expands. Moreover, after a few dynamical times the most important contribution to (5) will always come from the $m=1$ term. Thus, in the contracting direction the Wigner function approaches a Gaussian with a critical width σ_c and has the form

$$W(u,v,t) \approx \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-u^2/2\sigma_c^2} e^{-\omega t} \int_{-\infty}^{\infty} du\, W(u,v_0,t=0)\,. \tag{6}$$

The existence of the critical width σ_c is a consequence of the competition between the chaotic evolution (which attempts to "squeeze" the wave packet in the contracting direction) and the diffusion (which has the opposite tendency), and leads to a compromise steady state—the Gaussian written above.

How do these considerations lead to the three distinguishable stages of the evolution? The analysis of *de-*

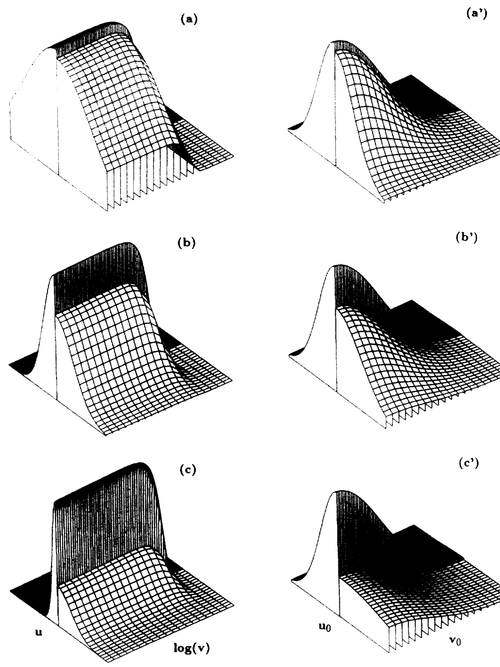

FIG. 1. Three stages of the evolution of an initially Gaussian Wigner function shown in two different coordinate systems. (a)-(c) W in physical coordinates (u, v, with v in logarithmic scale). For an open system which evolves according to Eq. (1) with $V=\lambda x^2/2$ (forefront), the width of the distribution reaches the asymptotic value σ_c. By contrast, when W evolves unitarily (densely hatched and shown in the back), it continues to be squeezed in u. (a')-(c') The same three stages of the evolution, but now in comoving coordinates [$\tilde{v}=v\exp(\lambda t)$, $\tilde{u}=u\exp(-\lambda t)$]. In the unitary case (cross-hatched, in the back) W does not change. The interaction with the environment causes an exponential increase in the apparent width of W (forefront). Since the width of the Gaussians in the expanding direction is approximately the same, the asymptotic regime of the diffusive evolution leads to an exponential increase of the area enclosed in 1σ contour. Consequently, the entropy increases linearly at a rate determined by the Lyapunov exponent.

coherence follows simply. Nonclassical states possessing a rapidly oscillating nonpositive W quickly evolve towards a mixture of localized states resulting in a smooth and positive Wigner function. For example, if the initial state is a superposition of two coherent states separated by a distance L (along u), the ratio between the wavelength of the interference fringes $l \propto 1/L$ and σ_c is $\sigma_c^2/l^2=(\gamma/\lambda)\times(L/\lambda_{\rm dB})^2$, where $\lambda_{\rm dB}$ is the thermal de Broglie wavelength. Therefore, the decoherence time is [8]

$$\tau_{\rm dec}=\gamma^{-1}(\lambda_{\rm dB}/L)^2\,, \tag{7}$$

which, for macroscopic scales, is much smaller than the dynamical times even for very weakly dissipative systems. Our analysis of the decoherence period is still incomplete since Eq. (6) suggests that the negativity of the Wigner function may persist along the expanding direction (in that equation the initial state is not changed along v but just "stretched" by the geodesic flow). However, Eq. (3) was obtained by neglecting gradients along v. When a diffusion term $D\partial_{vv}^2 W$ is added to the right hand side of (3), the eigenfunctions change in such a way that the powers $(ve^{-\lambda t})^n$ are replaced by Hermite polynomials $H_n(v/\sqrt{2}\sigma_c)e^{-n\lambda t}$. This implies that the asymptotic form of W is no longer given by (6), which just contains the initial state expanded along the unstable direction. The correct expression now contains a smoothed version of the initial condition in which the details smaller than σ_c are washed out along the unstable direction. Again, oscillations with wavelength $l \propto 1/L$ are destroyed after the decoherence time (7).

The analysis of the reversible and irreversible stages can be illustrated by following the evolution of a Gaussian W. Here, the existence of σ_c is again very important. The von Neumann entropy $\mathcal{H}$ of a Gaussian state can be easily related to the area A enclosed by a 1σ contour of the Wigner function. Thus $\mathcal{H}$ is a monotonic function of A which, when $A \gg h=2\pi\hbar$ approaches $\mathcal{H}\simeq k_B \ln A/h$ [10]. Using the above equations, one can show that the rate of entropy production is

$$\dot{\mathcal{H}}=\dot{A}/A=\lambda\sigma_c^2/\sigma_p^2(t)\,, \tag{8}$$

where $\sigma_p(t)$ is the width of the Gaussian along the direction of p. So when $\sigma_p(t)$ approaches the critical value, the entropy growth rate becomes equal to the positive Lyapunov exponent λ. Evolution of $\dot{\mathcal{H}}$ can be analyzed as follows: One can use (1) to show that the ratio $R\equiv[\sigma_p(t)/\sigma_c]^2$ evolves according to the equation $\dot{R}=2\lambda(1-\beta R)$ where $\beta(t)$, which is related to the degree of squeezing of the state, approaches unity exponentially fast. Solving this equation approximately (using $\beta=1$) one gets

$$\dot{\mathcal{H}}=\lambda\left[1+\left(\frac{\sigma_p^2(0)}{\sigma_c^2}-1\right)\exp(-2\lambda)\right]^{-1}. \tag{9}$$

Evolution of a classical distribution (corresponding to a non-negative W, initially smooth on a scale much larger than σ_c and spread over a regular patch with $A \gg h$) will typically proceed in two different stages (see Fig. 2). The first stage will be approximately area preserving with the evolution dominated by the Liouville operator. It will last as long as each of the dimensions of the patch is much larger than the critical width. During this stage diffusion does little to alter the form of W. The Wigner function is merely "stretched" or "contracted" by the geodesic flow so that, with respect to the comoving coordinates, "nothing happens" to W. By contrast, when the dimension of the patch becomes comparable with σ_c, diffusion will begin to dominate. Further contraction will be halted at σ_c but the stretching will proceed at the rate set by the positive Lyapunov exponent. As a result, the area (or, more generally, the volume) in phase space will increase at the rate set by (9) with $\sigma_p=\sigma_c$. Using our approximate Eq. (9) one can estimate the time corresponding to the transition from reversible to irreversible evolution:

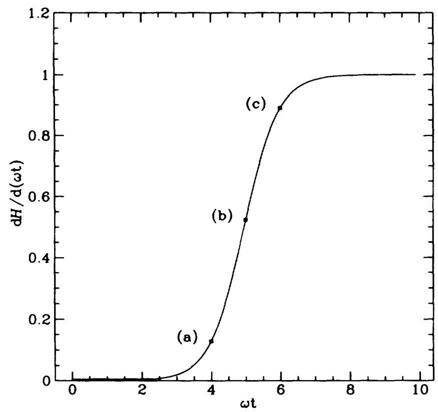

FIG. 2. The rate of von Neumann entropy production for the quantum open system. The initial state is a Gaussian for which $\mathcal{H}(t=0)\gg 1$ and with the initial width along the contracting direction much larger than σ_c. The (nearly) reversible and irreversible stages of the evolution are clearly distinguished.

$$\tau_c = \lambda^{-1}\ln[\sigma_p(0)/\sigma_c]\,. \tag{10}$$

To apply these arguments to more complicated nonlinear systems we need to verify that the Wigner function follows approximately the evolution generated by the Poisson bracket [i.e., that the effect of the higher derivative terms in (1) is small]. As we pointed out above, in the absence of decoherence the quantum corrections to the evolution of W become important at a crossover time t_χ given by Eq. (2). These corrections will remain small when diffusion is strong enough to prevent formation of small scale structure in W. Development of small scale structure induced by the exponential contraction is stopped after a time τ_c. Therefore, the condition for the Wigner function to evolve classically is

$$t_\chi \gg \tau_c\,, \tag{11}$$

which, taking into account Eqs. (2) and (10), can be rewritten in the following suggestive way reminiscent in form of the Heisenberg indeterminacy principle:

$$\chi_n \sigma_c \gg \hbar\,. \tag{12}$$

Thus, the scale of nonlinearities in the potential must be larger than the coherence scale $\hbar/\sigma_c$ on which the wave function of the system can maintain quantum coherence in spite of the coupling with the environment. This condition—a key criterion to ascertain the correspondence between quantum and classical dynamics—assures that W follows the Liouville flow (albeit with diffusive contributions).

We have demonstrated that chaotic quantum systems can exhibit, in addition to the very rapid onset of decoherence, a nearly reversible phase of evolution which is necessarily followed by an irreversible stage in which the entropy increases linearly at the rate determined by the Lyapunov exponents. By contrast, open quantum systems with regular classical analogs continue to evolve with little entropy production (although possibly with a significant change in dynamics [15,16]). This nearly constant rate of (von Neumann) entropy production, a consequence of the interplay between the chaotic dynamics of the system and its interaction with the environment, suggests not only a clear distinction between the integrable and chaotic systems, but also shows that increase of entropy in the context of quantum measurement and the dynamical aspects of the second law are intimately related and can be traced to the same cause: the impossibility of isolating macroscopic systems from their environments.

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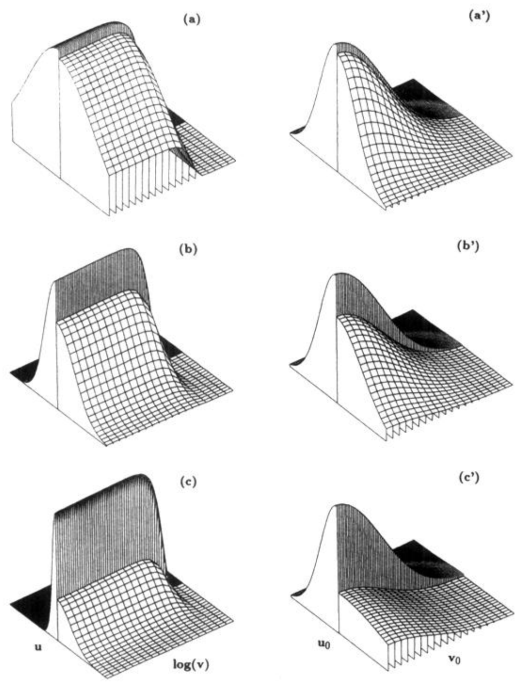

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